Quantum Operations

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Abstract

This article is a first look at the most general operations on quantum systems, named *quantum operations*. It is intended for students familiar with non-relativistic quantum mechanics including the density operator, product Hilbert spaces, and the Bloch sphere representation of two-state systems. The concept of quantum operations is defined and important theorems as well as examples are presented and explained. *Completely general quantum measurements* are introduced as a natural extension of measurements using the new concept of quantum operations. Advanced sections and footnotes are marked with an asterisk.

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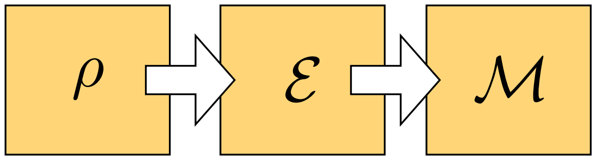
1 Introduction

1.1 Background

Experiments



A (quantum) experiment, in abstract terms, consists of preparing a system in a state, doing something to the system, and finally measuring something about the system. This procedure is then repeated on an ensemble of statistically independent systems. Measurement statistics are gathered and compared to theory.

Mathematical Description [1]

In non-relativistic quantum mechanics pure states of closed systems are represented by vectors in a Hilbert space of dimension d.	$\lvert\psi\rangle \in \mathcal{H}_d$
Linear operators map vectors to vectors.	$\lvert\psi\rangle \to \lvert\psi'\rangle = A\lvert\psi\rangle$
They have a matrix form.	$A_{ij} = \langle i\rvert A \lvert j\rangle$
The space of linear operators is called Liouville space.	$A \in \mathbb{L}_{d^2}$
Superoperators are linear maps of linear operators to linear operators.	$A \to A' = \mathcal{E}(A)$

Superoperators As stated above, superoperators are linear maps of linear operators to linear operators. If this is a new concept, you may wonder what mappings are included in the concept of superoperators. Here are some examples.

- $\mathcal{U}(A) = U\,A\,U^\dagger$
- $\mathcal{L}(A) = \frac{1}{\hbar}\,[H,\,A]_-$
- $\mathcal{T}(A) = A^T$

In general, a superoperator can be any linear map, i.e. any rank 4 tensor.

[1] *Hilbert spaces considered in this document are always assumed to have finite dimension. This is *not* a limitation of the quantum operation formalism. Infinite dimensions are eschewed here in order to avoid the required convergence arguments which distract from the conceptual understanding. They are handled in Kraus' lecture notes[2].

1.2 Motivation

Density Operators At this point, it is useful to recall the relationship of ket vectors to density operators. A closed system in a pure state can be described by a ket vector $|\psi\rangle$ in the Hilbert space $\mathcal{H}_d$. Density operators are generally introduced to describe statistical mixtures, however, there is source of mixtures. If our system of interest A is entangled with another system B, the reduced density matrix for A will in general be a mixture, even if the composite system AB is in a pure state. Furthermore, having access only to system A, there is no way to tell the two kinds of mixtures apart if they have identical density operators. (The expectation value of any operator O^A acting on A is completely determined by ρ^A: $\langle O^A \rangle_A = \mathrm{Tr}\left[O^A \rho^A\right]$.)

Statistical mixture	$\rho = \mathrm{p}_1 \|\psi_1\rangle \langle\psi_1\| + \mathrm{p}_2 \|\psi_2\rangle \langle\psi_2\|$
Subsystem mixture	$\rho^A = \mathrm{Tr}\left[\rho^{AB}\right], \rho^{AB} = {}_{AB}\langle\psi\| \|\psi\rangle_{AB}$

Operations? We have seen that density operators generalize pure states. But we are still acting on our system with "pure" operations, for example $\rho \to e^{-iHt/\hbar}\,\rho\, e^{+iHt/\hbar}$. How can we handle probabilistic operations [2] and operations involving open systems?

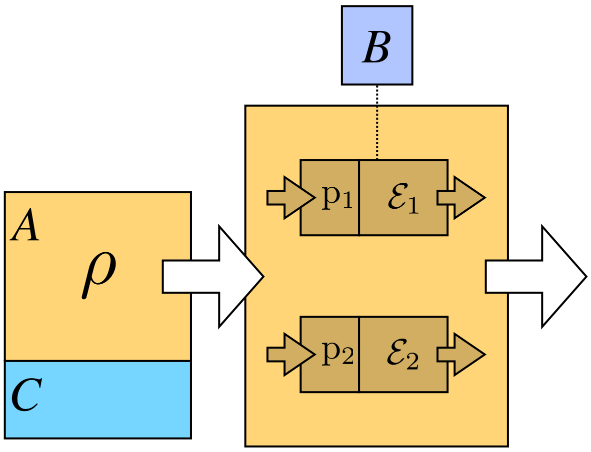


Figure 1: A complicated operation. A is initially entangled with C. $\mathcal{E}_1$ occurs with probability p_1 and $\mathcal{E}_2$ with probability p_2. $\mathcal{E}_1$ additionally entangles A with system B.

The Long Method We can already calculate the result of complicated operations, such as those discussed in the last section. Consider a system A entangled with a second system B. Unitary operators $U_{(i)}^{AB}$ act on the composite system.

[2]The reader may wonder why we would *want* to perform different operations on each system of the ensemble based on a random dice rolls. Indeed, this is a description of something not generally desired but always present: noise. The ability to describe the effects of noise on a system is a great strength of the quantum operation formalism.

The probability that $U^{AB}_{(i)}$ is carried out is given by p_i. The process for calculating the resulting density operator ρ'^{A} is straightforward:

1. Find ρ^{AB} . If A and B are initially separable, then $\rho^{AB} = \rho^{A} \otimes \rho^{B}$. Otherwise, we assume ρ^{AB} has been previously calculated or is otherwise known.

$$\text{Find } \rho^{AB}$$

2. Calculate the effect of each of the unitary operators on the composite system.

$$\rho'^{AB}_{(i)} = U^{AB}_{(i)}\,\rho^{AB} U^{AB\dagger}_{(i)}$$

3. Trace over the rest of the system to find the reduced density operator for A resulting from operation $U^{AB}_{(i)}$.

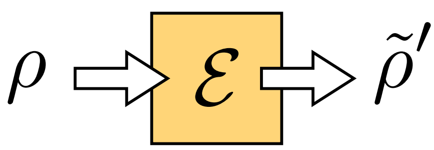
$$\rho'^{A}_{(i)} = \mathrm{Tr}_B\left[\rho'^{AB}_{(i)}\right]$$

4. Find the final density operator for A by weighting by the probabilities of each unitary operation.

$$\rho'^{A} = \sum_i \mathrm{p}_i \rho'^{A}_{(i)}$$

The above outlined "long" method produces the desired result without requiring a new mathematical formalism. In keeping with our analogy to density operators, note that density operators are also not absolutely required — we could meticulously keep track of all possible pure states and associated probabilities of the composite system. However, while not necessary, the density operator formalism streamlines these calculations, and additionally provides insight about what kinds of mixtures are distinguishable. We shall see analogous benefits from the quantum operation formalism.

The New Method In the "new" method, we are only concerned with system A. Other systems may be present, but they are only a *means* of acting on our system of interest A. We describe the action of all external entanglement and operations mathematically as: $\rho \rightarrow \tilde{\rho}' = \mathcal{E}(\rho)$ [3] or pictorially as:



This is called an in-out representation, in contrast to time dynamics using, for example, Schrödinger's equation. In this form, we are only concerned with the final state as a function of the initial state. [4]

[3] Tildes denote density operators that may not be properly normalized ($0 \leq \mathrm{Tr}\,[\tilde{\rho}'] \leq 1$).

[4] *Time evolution of the density operator of an open quantum system is given by a *Master Equation* using a *Lindbladian* with *Lindblad operators*. However, the Markovian assumption

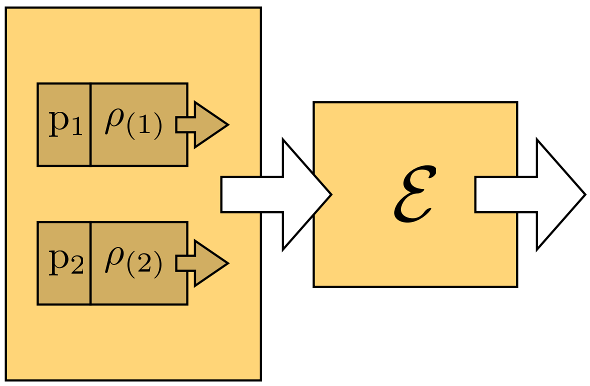
2 Quantum Operations

Quantum Operations are defined mathematically as mappings from density operators to density operators $\rho \rightarrow \tilde{\rho}' = \mathcal{E}(\rho)$ with the following three properties:

- Linear
- Does not increase the trace
- Completely positive

2.1 Properties

Linearity



Consider our operation applied to a statistically mixed input state. With probability p_1 our system is in the state $\rho_{(1)}$ and with probability p_2 our system is in the state $\rho_{(2)}$. When $\rho_{(1)}$ is input, $\mathcal{E}(\rho_{(1)})$ must be the result and similarly for $\rho_{(2)}$. The resulting density matrix is the weighted sum $p_1\mathcal{E}(\rho_{(1)}) + p_2\mathcal{E}(\rho_{(2)})$. But we can also write the (mixed) input density matrix as $p_1\rho_{(1)} + p_2\rho_{(2)}$ which has the output $\mathcal{E}(p_1\rho_{(1)} + p_2\rho_{(2)})$. We conclude:

$$\mathcal{E}(p_1\rho_{(1)} + p_2\rho_{(2)}) = p_1\mathcal{E}(\rho_{(1)}) + p_2\mathcal{E}(\rho_{(2)}) \tag{1}$$

This is exactly definition of linearity. Linearity implies that quantum operations are a subset of superoperators (recall that superoperators are linear maps from linear operators to linear operators).

Does not increase the trace Density operators have unity trace, so it might be expected that quantum measurements be maps from unity trace operators to unity trace operators. This requirement is relaxed slightly to allow *selective measurements* to fit into the formalism.

underpinning this approach does not in general hold for entangled systems meaning that the time evolution cannot be calculated without knowledge of the state of the composite system or alternatively complete knowledge of the history of the open system A. For more information, see [1, p277ff].

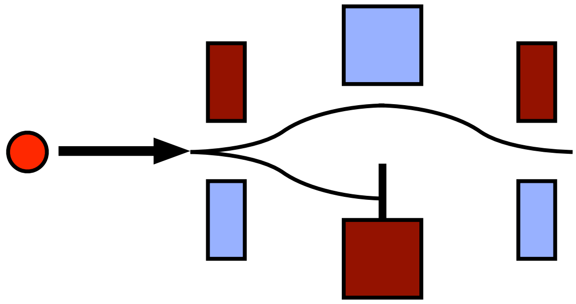


Figure 2: A selective measurement. Spin-down atoms are removed from the ensemble.

A *selective measurement* is shown in Figure 2. Our system, a spin-$\frac{1}{2}$ atom, moves through a Stern-Gerlach device. We choose to only use spin-up atoms and therefore block the lower path. This can be described as a quantum operation, but the resulting density operator will not be normalized, $\mathrm{Tr}\left[\mathcal{E}(\rho)\right] < 1$. For example, if half of the atoms are spin-down then $\mathrm{Tr}\left[\mathcal{E}(\rho)\right] = \frac{1}{2}$.

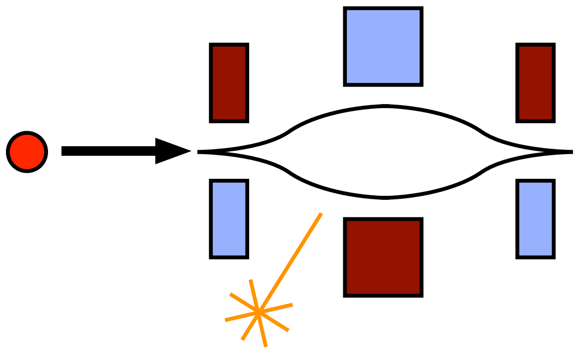


Figure 3: A non-selective measurement. Spin is measured, but no systems are removed from the ensemble.

A *non-selective measurement* is shown in Figure 3. A laser illuminates the center section where the spin-up and spin-down components of the atom are separated. For each atom we will see fluorescence in either the upper or lower path. After fluorescence, the spatially separated spin-up and spin-down systems are combined. If the measured spin information is not used to distinguish the systems [5] , the process can be described by a quantum operation. Because no systems were excluded, the resulting density operator will have unity trace, $\mathrm{Tr}\left[\mathcal{E}(\rho)\right] = 1$.

Considering selective measurements, we allow for systems being removed from

[5]If the information *is* used to distinguish the systems, we would have effectively split our initial ensemble and would therefore have two different density operators, one for each measurement result, i.e. $\tilde{\rho}'_{\uparrow}$ and $\tilde{\rho}'_{\downarrow}$. This would again be a selective measurement since we are selecting based on the measured value: $\mathrm{Tr}[\tilde{\rho}'_{\uparrow}] \leq 1$ and $\mathrm{Tr}[\tilde{\rho}'_{\downarrow}] \leq 1$.

the ensemble. Adding systems to the ensemble is, however, not possible. We therefore demand

$$\mathrm{Tr}\left[\mathcal{E}(\rho)\right] \leq 1 \quad \text{when} \quad \mathrm{Tr}\left[\rho\right] = 1 \tag{2}$$

If required, the resulting density operator can be normalized

$$\rho' = \frac{\mathcal{E}(\rho)}{\mathrm{Tr}\left[\mathcal{E}(\rho)\right]}$$

Completely positive The preceding argument for allowing the trace of the density operator to decrease was based on selective measurements removing systems from the ensemble. Such operations can at most remove *all* of the systems, in which case $\mathrm{Tr}\left[\tilde{\rho}'\right] = 0$. In other words, even unnormalized density operators must be positive operators. It seems logical to require $\mathcal{E}(\rho)$ to be a mapping from positive operators to positive operators, $\mathcal{E}(\rho) \geq 0$. It turns out this is not quite restrictive enough. [6]

The correct requirement follows from consideration of the possibility that system A is entangled with an external system B. The quantum operation acts *directly* only on A, but the entanglement may have an *indirect effect* on the entangled system B. [7]

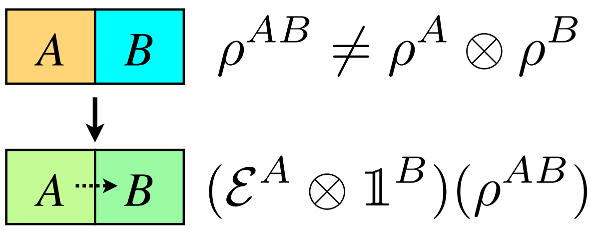


Figure 4: The operator $\mathcal{E}^A$ has an indirect effect on system B due to the existing entanglement between systems A and B. The resulting density operator for the composite system AB must be a positive operator.

The quantum operation for the combined system is written as $(\mathcal{E}^A \otimes \mathbb{1}^B)(\rho^{AB})$, understood to mean that $\mathcal{E}$ acts on the subspace $\mathcal{H}^A$ and $\mathbb{1}$ (i.e. nothing) acts on the subspace $\mathcal{H}^B$. We therefore require for quantum operations

$$(\mathcal{E}^A \otimes \mathbb{1}^B)(\rho^{AB}) \geq 0 \text{ for all dimensions of } \mathcal{H}^B \tag{3}$$

[6]*A superoperator which does not increase the trace and is positive but *not* completely positive is the transpose operator $\mathcal{T}(\rho) = \rho^T$ [1, p275].

[7]The prototypical example is an EPR experiment where observation of one system determines the state of the other, spatially-separated system.

2.2 Kraus' Theorem

The most important theorem relating to quantum operations is called Kraus' Theorem, or the Operator-Sum-Decomposition Theorem.

Theorem A mapping $\rho \to \tilde{\rho}' = \mathcal{E}(\rho)$ is a quantum operation (meaning it has the three properties linearity, non-increasing trace, and complete positivity) *if and only if* there exists an operator-sum decomposition

$$\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$$

with linear operators (called Kraus operators) that satisfy [8]

$$\sum_i K_i^\dagger K_i \leq \mathbb{1}$$

Remarks

- The inequality is strictly less-than when the quantum operation does not conserve the trace, and equal when the quantum operation conserves the trace.
- Kraus' theorem is an "if and only if" relation and therefore implies the existence of Kraus operators if the three properties 2 are fulfilled *and* guarantees the three properties for any set of Kraus operators.
- The number of Kraus operators is not constrained, and can even be infinite for finite dimensional $\mathcal{H}^A$.
- Kraus' theorem states only existence and not uniqueness. [9]

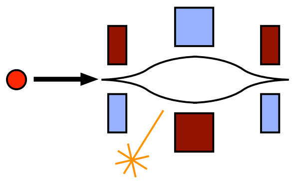
[8] *In the more general case of infinite dimensions, the inequality $\sum_i K_i^\dagger K_i \leq \mathbb{1}$ plays a key role in ensuring the proper definition and convergence of summations [2, p42ff].

[9] We will see in the examples that the set of Kraus operators describing a quantum operation is definitely *not* unique.

2.3 Simple Examples

We now consider two simple examples to familiarize the reader with quantum operations and the operator sum form.

Non-selective Measurement Our first example is the non-selective measurement discussed in 2.1. With probability $\langle 0_z| \rho |0_z\rangle$ the system is projected into the pure state $|0_z\rangle \langle 0_z|$ (and similarly for the 1_z direction).



$$\mathcal{E}_z(\rho) := \langle 0_z| \rho |0_z\rangle \; |0_z\rangle \langle 0_z| + \langle 1_z| \rho |1_z\rangle \; |1_z\rangle \langle 1_z|$$

This is nearly in the Kraus form. Since $\langle 0_z| \rho |0_z\rangle$ is a c-number, we may move it inside the associated projection operator $|0_z\rangle \langle 0_z|$.

$$\mathcal{E}_z(\rho) = |0_z\rangle \langle 0_z| \rho |0_z\rangle \langle 0_z| + |1_z\rangle \langle 1_z| \rho |1_z\rangle \langle 1_z|$$

Adding parentheses to guide the eyes, and taking the adjoint of the (self-adjoint) projection operators on the right,

$$\mathcal{E}_z(\rho) = (|0_z\rangle \langle 0_z|) \; \rho \; (|0_z\rangle \langle 0_z|)^\dagger + (|1_z\rangle \langle 1_z|) \; \rho \; (|1_z\rangle \langle 1_z|)^\dagger$$

This is now clearly in the Kraus form, implying that the operation is a quantum operation (is linear, does not increase the trace, and is completely positive). The Kraus operators can be read off:

$$K_1 = |0_z\rangle \langle 0_z| \qquad K_2 = |1_z\rangle \langle 1_z|$$

Something Different Our second example is of the new, probabilistic type of operation mentioned in the Motivation 1.2. The operation consists of doing nothing half of the time [10] , and half of the time applying σ_z (a rotation of π about the z-axis of the Bloch sphere). We can write its effect on a density operator as

$$\hat{\mathcal{E}}_z(\rho) := \frac{1}{2}\rho + \frac{1}{2}\sigma_z \, \rho \, \sigma_z$$

Splitting the $\frac{1}{2}$'s, adding unity operators, and taking the adjoint of self-adjoint operators where needed,

[10] An operation the author can identify with.

$$\hat{\mathcal{E}}_z(\rho) = \left(\frac{1}{\sqrt{2}}\mathbb{1}\right) \rho \left(\frac{1}{\sqrt{2}}\mathbb{1}\right)^\dagger + \left(\frac{1}{\sqrt{2}}\sigma_z\right) \rho \left(\frac{1}{\sqrt{2}}\sigma_z\right)^\dagger$$

Again, we have a Kraus form and therefore a quantum operation. This time the Kraus operators are:

$$\hat{K}_1 = \frac{1}{\sqrt{2}}\mathbb{1} \qquad \hat{K}_2 = \frac{1}{\sqrt{2}}\sigma_z$$

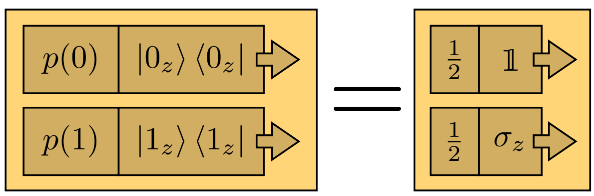
Non-uniqueness The effect of our quantum operations from the previous two examples can be calculated in matrix form for a general density operator. For the non-selective measurement we have:

$$\begin{aligned}
\mathcal{E}_z(\rho) &= \mathcal{E}_z\left(\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\right)\\
&= (|0_z\rangle\langle 0_z|)\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}(|0_z\rangle\langle 0_z|) + (1_z \text{ term})\\
&= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + (1_z \text{ term})\\
&= \begin{pmatrix}\rho_{11} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \rho_{22}\end{pmatrix}\\
&= \begin{pmatrix}\rho_{11} & 0\\ 0 & \rho_{22}\end{pmatrix}
\end{aligned}$$

And for the new, probabilistic quantum operation:

$$\begin{aligned}
\hat{\mathcal{E}}_z(\rho) &= \hat{\mathcal{E}}_z\left(\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\right)\\
&= \frac{1}{2}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\rho_{11} & -\rho_{12}\\ -\rho_{21} & \rho_{22}\end{pmatrix}\\
&= \begin{pmatrix}\rho_{11} & 0\\ 0 & \rho_{22}\end{pmatrix}
\end{aligned}$$

We conclude that the two sets of Kraus operators describe the same quantum operation.



Again, Kraus' Theorem only guarantees the existence of a Kraus operator decomposition, and says nothing about uniqueness. Non-uniqueness is in fact the general case. This is not surprising considering that there are many many ways to implement a given quantum operation, each of which has its own "natural" set of Kraus operators.

2.4 Quantum Channels

In classical information theory, information is moved from one place to another through information channels which have some (in general deleterious) effect on the information being passed through the channel.

Analogously, moving a quantum system from one location to another (or simply forward in time) can be viewed as moving the system through a *quantum channel*. The act of moving through the quantum channel has an effect on the system and the effect is a quantum operation.

Phase Damping Channel A generalization of the first example in 2.3 assigns the probability $\mathrm{p} \leq \frac{1}{2}$ to the σ_z operation and $1 - \mathrm{p}$ to doing nothing. [11]

$$K_1 = \sqrt{1 - \mathrm{p}}\mathbb{1} \qquad K_2 = \sqrt{\mathrm{p}}\sigma_z$$

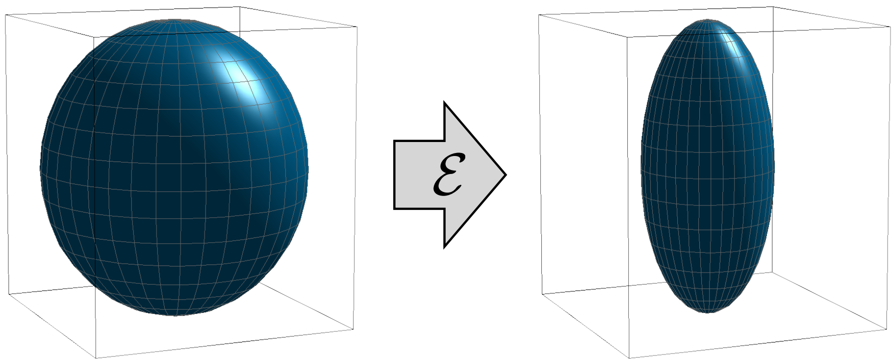


Figure 5: The effect of a phase damping channel (with $p = \frac{1}{4}$) on the Bloch sphere.

The effect of this quantum operation can be visualized by the mapping of points on the surface of the Bloch sphere (Figure 5). All points on the z-axis are fixed points and are mapped onto themselves by the operation. Repeated application of the quantum operation will collapse the entire Bloch sphere onto the z-axis. This is a model for noise that effects the phase of the Bloch vector.

Depolarization Channel The depolarization channel has a Kraus decomposition using the following four Kraus operators with $\mathrm{p} \leq \frac{3}{4}$. [12]

$$K_0 = \sqrt{1 - p}\,\mathbb{1} \qquad K_i = \sqrt{\frac{p}{3}}\,\sigma_i \quad (i = 1 \ldots 3)$$

[11] $\mathrm{p} > \frac{1}{2}$ also defines a quantum operation, one that inverts the phase of the Bloch sphere. However this operation cannot be realized by repeated application of a phase damping operation with $\mathrm{p} \ll \frac{1}{2}$ and is therefore not a model for phase damping noise.

[12] As in the previous example, $\mathrm{p} > \frac{3}{4}$ also defines a quantum operation, this time inverting the Bloch sphere. Again, this operation cannot be realized by repeated application of a depolarization channel with $\mathrm{p} \ll \frac{3}{4}$ and is therefore not a model for depolarization noise.

Points on the Bloch sphere move towards the center (Figure 6). Mixtures become more mixed. The completely mixed state at the origin of the Bloch sphere is a fixed point.

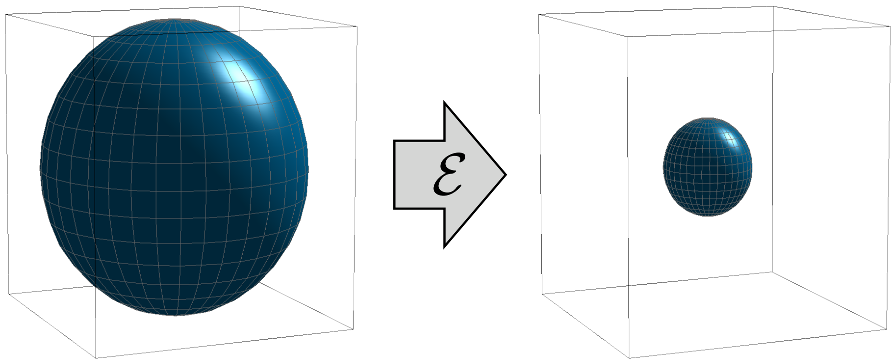


Figure 6: The effect of a depolarization channel ($p = \frac{1}{2}$) on the Bloch sphere.

Amplitude Damping Channel For the last example we will look at a simplified physical system and determine the Kraus operators. Our system is a two-level Atom A. A helper system, Box B [13] , initially contains zero photons. To be clear, the system B is prepared in the same empty state for each system A of the ensemble.

If the atom is initially in the ground state, there is no energy available and the atom will always exit the box in the ground state. If the atom is in the excited state, there is a probability p that the atom emits a photon into the box (see Figure 7).

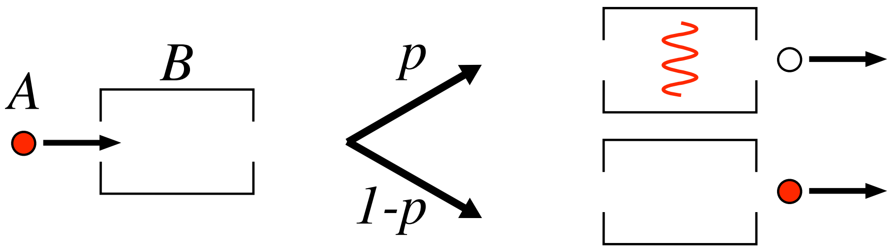


Figure 7: An atom A in the excited state has probability p to emit a photon when passing through the initially empty box B.

The probabilities determine the associated matrix elements of the transformation matrix U^{AB}. [14]

[13] Actually not a "box" but a resonate cavity, which unfortunately does not begin with B.

[14] We use the pictorial basis vectors $|\bigcirc\rangle_A$, $|\bullet\rangle_A$ for the atom in the ground and exited state and $|\sqsubset\rangle_B$, $|\sqsubset\rangle_B$ for the box with zero and one photon respectively.

$$\begin{aligned}
\left| \langle \bigcirc, \boxdot |\, U^{AB}\, | \bullet, \square \rangle \right|^2 &= p \\
\left| \langle \bullet, \square |\, U^{AB}\, | \bullet, \square \rangle \right|^2 &= 1 - p \\
\left| \langle \bigcirc, \square |\, U^{AB}\, | \bigcirc, \square \rangle \right|^2 &= 1
\end{aligned}$$

All other matrix elements with $|\square\rangle_B$ in the initial state vanish. We can now determine the effect of the operation using the "long" method (1.2) with our initial state $\rho^A \otimes |\square\rangle_B\ {}_B\langle\square|$.

$$\begin{aligned}
\rho'^A &= \mathrm{Tr}_B \left[U^{AB} \left(\rho^A \otimes |\square\rangle_B\ {}_B\langle\square| \right) U^{AB\dagger} \right] \\
&= \mathrm{Tr}_B \left[U^{AB} |\square\rangle_B\, \rho^A\ {}_B\langle\square|\, U^{AB\dagger} \right] \\
&= {}_B\langle\square|\, U^{AB} |\square\rangle_B\, \rho^A\ {}_B\langle\square|\, U^{AB\dagger} |\square\rangle_B \; + \; {}_B\langle\boxdot|\, U^{AB} |\square\rangle_B\, \rho^A\ {}_B\langle\square|\, U^{AB\dagger} |\boxdot\rangle_B
\end{aligned}$$

By explicitly taking the trace over $\mathcal{H}^B$ we have found the Kraus operators. They are ${}_B\langle\square|\, U^{AB} |\square\rangle_B$ and ${}_B\langle\boxdot|\, U^{AB} |\square\rangle_B$, sub-matrices of the transition matrix U^{AB}. In fact they are the first column of sub-matrices. Ignoring possible phases [15], the Kraus operators can be written in the $|\bigcirc\rangle_A = \binom{1}{0}$, $|\bullet\rangle_A = \binom{0}{1}$ basis using the matrix elements found above.

$$K_0^A = {}_B\langle\square|\, U^{AB} |\square\rangle_B = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad K_1^A = {}_B\langle\boxdot|\, U^{AB} |\square\rangle_B = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

The action on the Bloch sphere (Figure 8) is a compression towards the top of the sphere. The atom ground state at the top of the Bloch sphere is a fixed point. This is what we would expect given the physical system we started with. The empty box provides a means for excited-state atoms to enter the ground state, but no means to enter the excited state. Any mixture passing through this channel must evolve in the direction of the pure ground state.

[15]*Arbitrary phases in the elements of the Kraus operators cannot be ruled out from the physical description (in terms of probabilities) of this example. A phase of K_1^A would cancel in the Kraus sum, but a phase difference between the two non-zero components of K_0^A is relevant. Its effect would be an additional overall rotation of the Bloch sphere about the z-axis.

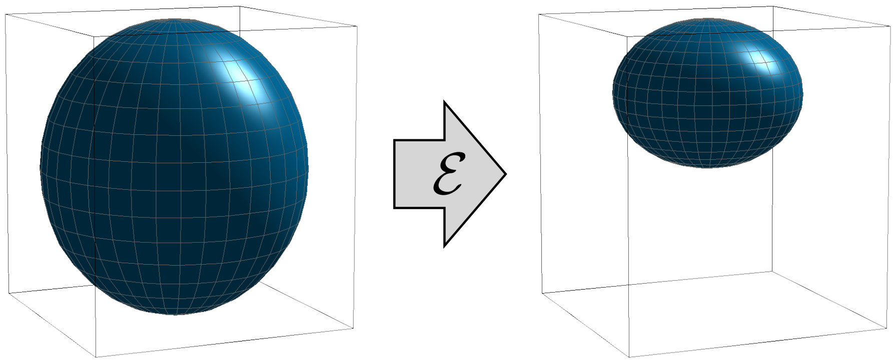


Figure 8: The effect of an amplitude damping channel ($p = \frac{1}{2}$) on the Bloch sphere.

2.5 Unitary Implementation Theorem

Quantum operations can be implemented in many ways. Interestingly, they can always be implemented using a helper system with unitary operations and projections.

Theorem For every quantum operation, there exists a helper system B initially in a pure state, a unitary operator U^{AB} acting on the composite system AB, and a projection operator P^B acting on the helper system B such that the operation of U^{AB} followed by P^B is a realization of the quantum operation.

$$\mathcal{E}(\rho^A) = \mathrm{Tr}_B \left[P^B U^{AB} \left(\rho^A \otimes |1\rangle_B \, {}_B\langle 1| \right) U^{AB\dagger} P^{B\dagger} \right]$$

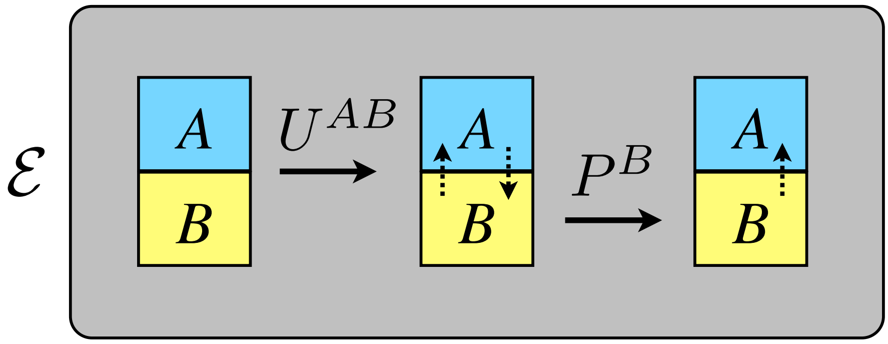


Figure 9: A unitary implementation of $\mathcal{E}$. The unitary operator U^{AB} mixes the initially separable systems A and B. Then, a projection operator acting on B has an indirect effect (through the entanglement) on A.

Proof Let $\{K_i\,;\, i = 1 \ldots N\}$ be a set of Kraus operators for the quantum operation. Define an additional Kraus operator K_{N+1} such that $K_{N+1}^{A\dagger} K_{N+1}^{A} =$

$\mathbb{1} - \sum_i^N K_i^{A\dagger} K_i^A$. Define a helper system $\mathcal{H}^B$ with dimension $N+1$. Let $\{|i\rangle_B \; ; \, i = 1 \ldots N+1\}$ be an orthonormal basis for $\mathcal{H}^B$ with $|1\rangle_B$ the initial pure state of system B. Define a unitary operator U^{AB} on $\mathcal{H}^{AB}$ such that ${}_B\langle i|\, U^{AB} |1\rangle_B = K_i^A$. [16] Define a projective operator P^B on $\mathcal{H}^B$ as $P^B = \mathbb{1} - |N+1\rangle_B \, {}_B\langle N+1|$. The proof now follows from evaluating $\tilde{\rho}'^A$.

$$
\begin{aligned}
\tilde{\rho}'^A &= \mathrm{Tr}_B \left[P^B U^{AB} \left(\rho^A \otimes |1\rangle_B \, {}_B\langle 1| \right) U^{AB\dagger} P^{B\dagger} \right] \\
&= \sum\nolimits_i^{N+1} {}_B\langle i|\, P^B U^{AB} |1\rangle_B \, \rho^A \, {}_B\langle 1|\, U^{AB\dagger} P^{B\dagger} \, |i\rangle \\
&= \sum\nolimits_i^{N} {}_B\langle i|\, U^{AB} |1\rangle_B \, \rho^A \, {}_B\langle 1|\, U^{AB\dagger} \, |i\rangle \\
&= \sum\nolimits_i^{N} K_i^A \rho^A K_i^{A\dagger} \\
&\square
\end{aligned}
$$

[16] The columns of a unitary operator form a set of orthonormal vectors. Finding a unitary operator given the requirement ${}_B\langle i|\, U^{AB} |1\rangle_B = K_i^A$ is equivalent to completing a partial set of orthonormal vectors which can always be done.

3 Completely General Selective Measurements

Review: Projective Measurements Projective measurements consist of a set of possible measurement values $\{m\}$ and associated projection operators P_m which fulfill $\sum_m P_m = \mathbb{1}$. The measurement value which actually occurs is non-deterministic, with probabilities $\mathrm{p}(m) = \mathrm{Tr}\left[P_m \rho P_m^\dagger\right]$. If value m is measured, the new state is transformed as

$$\rho \to \tilde{\rho}' = P_m \rho P_m^\dagger$$

Completely General Selective Measurements The generalization of projective measurements is to replace the projection operators with quantum operations $\mathcal{M}_m$.

If the value m is measured, the state is transformed as

$$\rho \to \tilde{\rho}' = \mathcal{M}_m(\rho) = \sum_i M_{m,i} \rho M_{m,i}^\dagger$$

Where the operators $M_{m,i}$ are the Kraus operators for each possible measurement outcome m. The number of operators is in general different for each possible measurement outcome (i.e. the range of i can be different for each m). The probability of a particular measurement outcome m is

$$\mathrm{p}(m) = \mathrm{Tr}\left[\mathcal{M}_m(\rho)\right]$$

Some value is always measured (if necessary, by defining a null measurement value), which implies that the sum of the probabilities is one, or alternatively, that the set of Kraus operators over all measurement values is a complete measurement.

$$\sum_m \mathrm{p}(m) = \sum_m \mathrm{Tr}\left[\mathcal{M}_m(\rho)\right] = 1 \quad \Longleftrightarrow \quad \sum_{m,i} M_{m,i}^\dagger M_{m,i} = \mathbb{1}$$

3.1 Unitary Implementation Theorem

Similar to quantum operations, completely general selective measurements can be implemented with a helper system using the familiar unitary operations and projective measurements.

Theorem [17] For every completely general selective measurement, there exists a helper system B initially in a pure state, a unitary operator U^{AB} acting on the composite system AB, and a set of projective measurement operators P_m^B acting on the helper system B such that the operation of U^{AB} followed by The projective measurement with P_m^B is a realization of the completely general selective measurement.

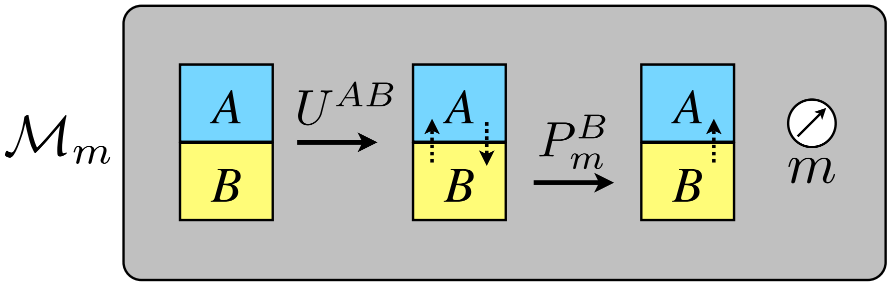
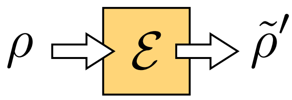


Figure 10: A unitary implementation of a completely general quantum measurement. The unitary operator U^{AB} mixes the initially separable systems A and B. Then, a projective measurement made on B yields a the measurement outcome m and has an indirect effect (through the entanglement) on A.

[17]The (omitted) proof of this theorem uses the same techniques used for the proof of the unitary implementation of quantum operations.

A Summary of Generalized Quantum Mechanics

Quantum Operations



A quantum operation $\mathcal{E}$ describes the deterministic transformation of an open quantum system. [18]

$$\rho \rightarrow \tilde{\rho}' = \mathcal{E}(\rho)$$

It has a Kraus sum decomposition.

$$\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$$

If the Kraus operators sum to the identity operator, the operation is *complete*, otherwise it is an *incomplete* operation.

$$\sum_i K_i^\dagger K_i \leq \mathbb{1}$$

Completely General Selective Measurements

Completely general selective measurements produce a measurement value and a non-deterministic state evolution. The evolution for a particular measurement value is itself a quantum operation.

$$\rho \rightarrow \tilde{\rho}' = \mathcal{M}_m(\rho) \qquad \mathrm{p}\,(m) = \mathrm{Tr}\,[\mathcal{M}_m(\rho)] \qquad \sum_m \mathrm{p}\,(m) = 1$$

The $\mathcal{M}_m$ operators have a Kraus representation. The Kraus operators for all possible measurement values together form a complete operation.

$$\mathcal{M}_m(\rho) = \sum_i M_{m,i} \rho M_{m,i}^\dagger \quad \mathrm{p}\,(m) = \mathrm{Tr}\left[\sum_i M_{m,i} \rho M_{m,i}^\dagger\right] \quad \sum_{m,i} M_{m,i}^\dagger M_{m,i} = \mathbb{1}$$

[18] A non-selective measurement has a deterministic effect and is therefore also considered a quantum operation.

B *Proof of Kraus' Theorem

Proving Kraus' Theorem requires proving both directions. We start with the easy one.

Kraus 1 Given a set of *Kraus operators* $\{K_i \,;\, i = 1 \ldots N\}$ satisfying the inequality $\sum_i^N K_i^\dagger K_i \leq \mathbb{1}$, the operation $\mathcal{E}(\rho) = \sum_i^N K_i \, \rho \, K_i^\dagger$ is a *quantum operation.*

Proof Linearity and not increasing the trace follow from the Kraus operator sum form and the requirement $\sum_i^N K_i^\dagger K_i \leq \mathbb{1}$. We consider complete positivity by evaluating ${}_{AB}\langle\psi| \, \tilde{\rho}'^{AB} |\psi\rangle_{AB}$ for arbitrary ρ^{AB} and $|\psi\rangle_{AB}$. Let d' be the dimension of the product Hilbert space $\mathcal{H}^{AB}$. Let $\{|j\rangle_{AB} \,;\, j = 1 \ldots d'\}$ be the eigenvectors and $\{\mathrm{p}_j \,;\, j = 1 \ldots d'\}$ the non-negative eigenvalues of the initial density operator ρ^{AB}. Then,

$$
\begin{aligned}
{}_{AB}\langle\psi| \, \tilde{\rho}'^{AB} |\psi\rangle_{AB} &= \sum_i {}_{AB}\langle\psi| \, K_i^A \rho^{AB} K_i^{A\dagger} |\psi\rangle_{AB} \\
&= \sum_{i,j} {}_{AB}\langle\psi| \, K_i^A \Big(\mathrm{p}_j |j\rangle_{AB} \; {}_{AB}\langle j| \Big) K_i^{A\dagger} |\psi\rangle_{AB} \\
&= \sum_{i,j} \mathrm{p}_j \Big| \; {}_{AB}\langle\psi| \, K_i^A |j\rangle_{AB} \; \Big|^2 \\
&\geq 0 \\
&\square
\end{aligned}
$$

Kraus 2 Given a *quantum operation* $\mathcal{E}(\rho)$, there exists a set of *Kraus operators* $\{K_i \,;\, i = 1 \ldots N\}$ such that $\sum_i^N K_i^\dagger K_i \leq \mathbb{1}$ and $\mathcal{E}(\rho) = \sum_i^N K_i \, \rho \, K_i^\dagger$.

Proof [19] Let d be the dimension of $\mathcal{H}^A$. Let $\mathcal{H}^B$ be a second system of dimension d. [20] Let $\{|i\rangle_A \,;\, i = 1 \ldots d\}$ be an orthonormal basis of $\mathcal{H}^A$ and $\{|i\rangle_B \,;\, i = 1 \ldots d\}$ be an orthonormal basis of $\mathcal{H}^B$.

We will characterize the operation of $\mathcal{E}$ on $\mathcal{H}^A$ completely by the operation of the extended operator $(\mathcal{E}^A \otimes \mathbb{1}^B)$ on a specific maximally entangled (and unconventionally normalized [21]) state $|\tilde{\Psi}\rangle_{AB}$ defined as

$$
|\tilde{\Psi}\rangle_{AB} := \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B
$$

[19] The proof employs the techniques used by Audretsch [1, p308ff] and Preskill [4, p100ff].

[20] A subtle point: only positivity with $\dim(\mathcal{H}^B) = \dim(\mathcal{H}^A)$ is required to prove a quantum operation has a Kraus sum representation. Considering $\mathcal{H}^B$ of lesser dimension is insufficient. Positivity with $\dim(\mathcal{H}^B) > d$ does not provide further restrictions.

[21] Using ${}_{AB}\langle\tilde{\Psi}|\tilde{\Psi}\rangle_{AB} = d$ instead of 1 avoids many factors of $\sqrt{d}$ which would simply cancel in the end.

Because $\mathcal{E}$ is completely positive, the operation of $(\mathcal{E}^A \otimes \mathbb{1}^B)$ on the density matrix $|\tilde{\Psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\Psi}|$ is also an (unconventionally normalized) density operator, which we write in its eigenbasis as

$$\left(\mathcal{E}^A \otimes \mathbb{1}^B\right)\left(|\tilde{\Psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\Psi}|\right) = \sum_j q_j |\Phi_j\rangle_{AB}\ {}_{AB}\langle\Phi_j| \,\text{with}\ q_j \geq 0,\ \sum_j q_j \leq d$$

Define the (anti-linear) mapping $|\psi\rangle_A \rightarrow |\psi^*\rangle_B$ from $\mathcal{H}^A$ to $\mathcal{H}^B$ by the conjugation of the coefficients of our chosen basis vectors. In the literature, $|\psi\rangle_A$ is called the "relative state" of the "index state" $|\psi^*\rangle_B$.

$$|\psi\rangle_A = \sum_i c_i |i\rangle_A \longrightarrow |\psi^*\rangle_B = \sum_i c_i^* |i\rangle_B$$

Note that the anti-linear mapping and maximally entangled state are constructed to have the following property:

$$|\psi\rangle_A = {}_B\langle\psi^*|\tilde{\Psi}\rangle_{AB}$$

Lastly, define the Kraus operators as the following mapping. This is linear because the $|\psi\rangle_A \rightarrow |\psi^*\rangle_B$ mapping is anti-linear.

$$K_j^A |\psi\rangle_A \longrightarrow \sqrt{q_j}\ {}_B\langle\psi^*|\Phi_j\rangle_{AB}$$

We are now ready to apply $\mathcal{E}$ to a general density matrix of $\mathcal{H}^A$ and see that it has the Kraus sum decomposition.

$$\begin{aligned}
\mathcal{E}^A\left(\rho^A\right) &= \mathcal{E}^A\Big(\sum_i \mathrm{p}_i |\psi_i\rangle_A\ {}_A\langle\psi_i|\Big) \\
&= \sum_i \mathrm{p}_i \mathcal{E}^A\Big(|\psi_i\rangle_A\ {}_A\langle\psi_i|\Big) \\
&= \sum_i \mathrm{p}_i \mathcal{E}^A\Big(\ {}_B\langle\psi_i^*|\tilde{\Psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\Psi}|\psi_i^*\rangle_B\Big) \\
&= \sum_i \mathrm{p}_i\ {}_B\langle\psi_i^*|\, \mathcal{E}^A\Big(|\tilde{\Psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\Psi}|\Big)|\psi_i^*\rangle_B \\
&= \sum_i \mathrm{p}_i\ {}_B\langle\psi_i^*|\Big(\sum_j q_j |\Phi_j\rangle_{AB}\ {}_{AB}\langle\Phi_j|\Big)|\psi_i^*\rangle_B \\
&= \sum_{i,j} \mathrm{p}_i q_j \Big(\ {}_B\langle\psi_i^*|\Phi_j\rangle_{AB}\ {}_{AB}\langle\Phi_j|\psi_i^*\rangle_B\Big) \\
&= \sum_{i,j} \mathrm{p}_i \Big(K_j^A |\psi_i\rangle_A\ {}_A\langle\psi_i|\, K_j^{A\dagger}\Big) \\
&= \sum_j K_j^A \Big(\sum_i \mathrm{p}_i |\psi_i\rangle_A\ {}_A\langle\psi_i|\Big) K_j^{A\dagger} \\
&= \sum_j K_j^A \rho^A K_j^{A\dagger}
\end{aligned}$$

□

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