Superconducting Quantum Bits

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Abstract

In this paper an important class of quantum bits, namely the superconducting quantum bits will be introduced. First a few basic principles will be discussed, among others the quantum mechanical behavior of the macroscopic degrees of freedom in superconductors. The Josephson Junction, a very important component allowing superconducting quantum bits, will also be described, and the physical processes behind it will also be looked at. Finally, two basic examples of superconducting quantum bits: the Cooper pair box, or charge quantum bit, or the rf-SQUID, or flux quantum bit.

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1 Introduction

An important part of quantum computing theory is the quantum bit (qubit). A qubit is an abstract mathematical concept that can be realized by physical systems. Common examples of systems that can be used as qubits are the polarization of light and spin-$\frac{1}{2}$ qubits. Important for these systems is that they have some degree of freedom that behaves quantum mechanically. In the two examples mentioned above, these degrees of freedom are strictly microscopic in nature: spin, obviously, of electrons and atoms; or the polarization of light. There are, however, possibilities of realizing a qubit system with *macroscopic* degrees of freedom.

This new class of qubit is based on the fact that some macroscopic degrees of freedom in superconducting materials behave quantum mechanically. As a result, the qubits described by this class are called superconducting (SC) qubits. SC qubits are in fact simply SC circuits.

1.1 Requirements for Qubit Implementation

Before introducing SC qubits, it is important to first understand how superconductors fulfill the requirements for qubit implementation.

The first requirement is that the degrees of freedom must behave quantum mechanically. In order for this to happen, the different possible states must retain phase coherence. This is partially realized with superconductors in that they transport electrons without energy loss (non-dissipative transport).

The second requirement is that the qubit be non-linear. This is important because linear systems exhibit harmonic behavior. Figure 1 on the following page shows the difference between a harmonic and an anharmonic oscillator. In the case of the harmonic oscillator there is no guarantee of qubit preservation. If both the ground state and first excited states are populated, any effort to move to a system where only the first state is populated will result in the loss of information. Because the energy transitions are degenerate, the energy to move from the ground state to the first state is the same as that for a transition from the first state to the second. The anharmonic oscillator, however, isolates the to lowest states. This means that the energy transitions are non-degenerate, ensuring that no information is lost when populating the first state.

Both of these requirements are met by devices called Josephson Junctions. These will be introduced at the end of a quick introduction to superconductivity.

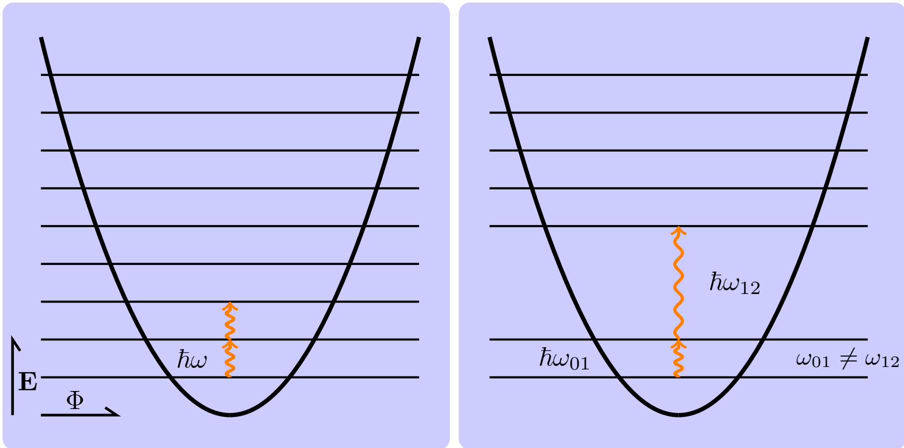

Figure 1: Pictured on the left is an example of harmonic behavior. Energy transitions are On the right is a situation where a non-linear element is introduced. Two energy levels are isolated.

2 Superconductivity and Quantum Mechanics

Both the theories of superconductivity and quantum mechanics have been around for a long time. It was not until the 1980's, however, that research was undertaken to see if the laws of quantum mechanics applied to macroscopic systems. Researchers found a number of macroscopic phenomena that followed the laws of quantum mechanics and exhibit quantum behavior, such as quantum tunneling. At the end of the 1990's researchers began to realize that Josephson devices could be used as qubits.

2.1 Superconductivity

One of the main theories describing the behavior of SC materials is the so-called BCS theory, named after its discoverers Bardeen, Cooper and Schrieffer. This theory is a microscopic theory that says the units of the supercurrent, the current flowing through a superconductor, are Cooper pairs. The are pairs of electrons bound together, resulting in a charge of $2e$, a mass of $2m_e$ and a total spin of zero, in other words, they are bosons. Furthermore, all of these Cooper pairs are, in the words of Clarke *et al.*, "condensed into a single macroscopic state described by a wavefunction"

$$\Psi(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)| \exp^{i\phi(\mathbf{r}, t)}$$

where $\mathbf{r}$ is the position and t is the time. The phase $\phi(\mathbf{r}, t)$ is an order parameter of the SC material, which means that it is unique to the material. This wave function leads us to two important phenomena for qubits: flux quantization, and Josephson tunneling.

2.1.1 Flux Quantization

The flux quantization can be seen after undertaking the typical process to create a SC current in a loop. A SC ring cooled below its critical temperature (the temperature below which superconductivity is seen) while in a magnetic field will cause a super current to flow through the ring if is removed from the magnetic field. This mechanism ensures that the magnetic flux through the ring remains constant. This is a surprising effect in itself, even more surprising is that this flux is quantized with integer values of

$$\Phi_0 = \frac{h}{2e} \approx 2.07 \times 10^{-15}\mathrm{Tm}^2$$

where h is Planck's constant, and e is the elementary charge.

2.2 The Josephson Junction

The Josephson Junction (JJ) is the device which makes SC qubits possible. A

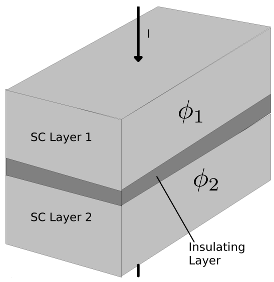

Figure 2: Basic representation of a Josephson Junction (JJ).

very basic representation of a JJ is depicted in figure 2. The JJ consists of an insulating or even metallic, non-SC layer sandwiched by two SC layers. This non-SC layer is on the oder of a few atoms thick. As discussed earlier, the two SC layers can be characterized by their phases, which are the order parameters of each side. The difference of these two parameters is the phase difference, δ

$$\delta = \phi_1 - \phi_2$$

In figure 3 on the following page the equivalent circuit of the JJ is depicted. It is important to note that there are two different electrical components that make up the JJ. The first is the capacitor, characterized by its capacitance C_J. The second is the "bare" Josephson element acts like an inductor with the inductance $L_{J0} = \frac{\Phi_0}{2\pi I_C}$ (also known as the effective Josephson inductance).

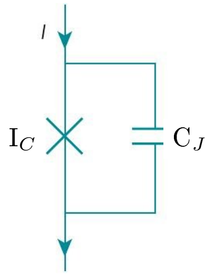

Figure 3: The equivalent circuit of the JJ [2]

2.2.1 The Josephson Equations

With two very important equations we can describe the behavior of the JJ. The first describes the behavior of the current through the JJ:

$$I = I_C \sin \delta \tag{1}$$

There are a couple of interesting effects here. First, one sees that despite the non-SC layer being present *there is still a supercurrent flowing though the junction.* This is incredible, because according to BCS theory, it should not be possible for Cooper pairs to be present in this non-SC layer. This means that the Cooper pairs are *tunneling* through the non-SC layer, a quantum mechanical effect.

Second, one can clearly see the meaning of the critical current I_C. The critical current is the *maximal* current that can flow through the junction. This is determined by the cross-section of the junction and the width of the non-SC layer.

Finally, one should pay close attention to what is missing. If one thinks back to classical circuit theory, the relationship between current and voltage is given by Ohm's law:

$$V = RI$$

As one can see here, the current does not have a direct voltage dependence. Instead the current is periodically dependant on the phase difference of the two SC layers.

The second equation describing the behavior of a JJ has to do with the voltage:

$$V = \frac{\hbar \dot{\delta}}{2e} \tag{2}$$

Solving this equation for $\dot{\delta}$, integrating over time and plugging back into equation (1) we see an interesting effect. Applying a dc-voltage across the junction results in an ac-current. This again is quite astonishing behavior.

2.2.2 Energy of the Josephson Junction

As seen in the last section, there are two different effects in the JJ, each making its own contribution to the total energy of the JJ.

First, there is the contribution from the Josephson capacitance, C_J. From circuit theory we know that the energy stored in a circuit component can be calculated by taking the time integral over the product of the current through and voltage across that element, i.e.

$$E = \int I \cdot V dt \tag{3}$$

In the case of the Josephson capacitance, where $V = \frac{Q}{C_J}$, $I = \dot{Q}$ and $Q = 2e$ (a single Cooper pair) we get:

$$\begin{aligned} E_{C_J} =& \int \dot{Q} \cdot \frac{Q}{C_J} dt \\ =& \frac{1}{C_J} \int Q dQ \\ =& \frac{Q^2}{2C_J} \\ =& \frac{2e^2}{C_J} \end{aligned} \tag{4}$$

This value scales with n^2, the number of Cooper pairs present in the capacitor squared.

The second element, the bare Josephson element, which has the effective Josephson inductance L_{J0} and provides us with the effective Josephson coupling energy.

$$E_J = \frac{I_C \Phi_0}{2\pi}.$$

The Josephson coupling energy, once again, can be calculated using equation (3) with the Josephson equations (1) and (2):

$$\begin{aligned} E_{coup} =& \int I \cdot V dt \\ =& \int I_C \sin\delta \cdot \frac{\hbar\dot{\delta}}{2e} dt \\ =& \frac{I_C \cdot h}{2\pi \cdot 2e} \int \sin\delta d\delta \\ =& -E_J \cos\delta \end{aligned} \tag{5}$$

With this knowledge it is now possible to describe a SC qubit.

3 Basic Superconducting Qubits

There are two basic types of SC qubits that will be examined here. The first SC qubit to be examined is the Cooper pair box, or charge qubit and is pictured in figure 4.

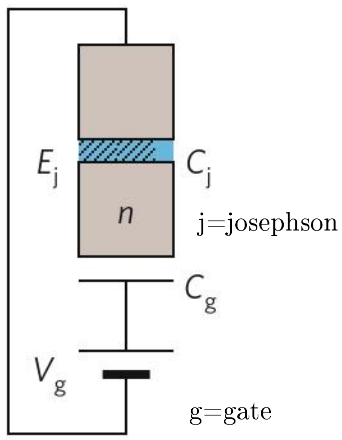

Figure 4: The Cooper pair box, or charge qubit [4].

The second type of SC qubit is the flux qubit, which is also an rf-SQUID, and it is pictured in figure 8 on page 11.

3.1 The Cooper Pair Box

The Cooper pair box consists of a Josephson junction connected with a bias voltage, V_g and a gate capacitance C_g, making the total capacitance of the qubit $C_\Sigma = C_J + C_g$. It is important that the SC island between the capacitors be small enough (and thus the capacitance also small enough) so that single-electron charging energy of the qubit E_{C_J} be much larger than the thermal energy $E_{\text{th}} = k_B T$. This minimizes any disturbing thermal effects.

The qubit is characterized by n, the number of Cooper pairs on the island, and choose the qubit charge states to be $|n\rangle$ and $|n+1\rangle$. One might quickly jump to the conclusion that the charging energy of the total capacitance in this qubit might be given by

$$E_{ch} = \underbrace{\frac{2e^2}{C_\Sigma}}_{=E_C} n^2$$

however, this is wrong. The mechanism for controlling the qubit needs to be taken into account. In this case, the qubit is controlled through the gate voltage, V_g, which results in a gate charge $Q_g = C_g V_g$. The dimensionless gate charge is then $n_g = \frac{C_g V_g}{2e}$, which finally results in the correct term for the charging energy of the qubit

$$E_{ch} = E_C \left(n - n_g\right)^2$$

It is important to note that although n is an integer, n_g is a continuous variable.

It is now necessary to look at this term quantum mechanically. In switching over to quantum mechanics, the variable n transforms into an operator $\widehat{n}$. The energy term the becomes an unperturbed (some might say "unpertabated") Hamiltonian:

$$\widehat{H}_0 \left|n\right\rangle = E_C \left(n - n_g\right)^2 \left|n\right\rangle$$

A plot of the eigenfunctions of this Hamiltonian is given in figure 5.

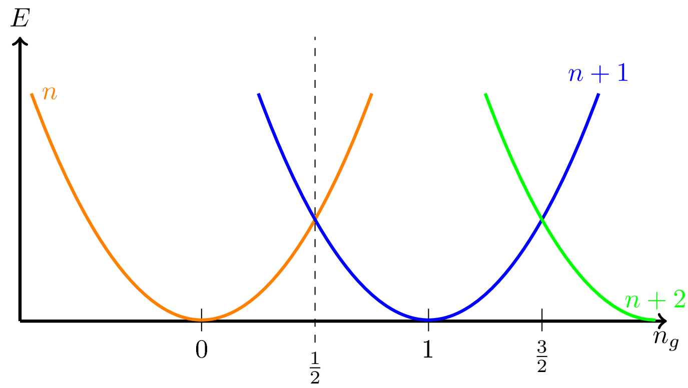

Figure 5: The energy of the Cooper pair box capacitance plotted for n and n_g. For simplicity sake we take $n = 0$ and $n + 1 = 1$.

As one can see, there are degeneracy points for the odd-multiples of $1/2$. This means that at these points there is a virtual charge on the SC island equal to one half of a Cooper pair.

This, however, is not the complete story of the qubit. The second energy term, which has not yet been taken into account, acts as a perturbation. This term comes from quantizing the Josephson coupling energy:

$$\delta \rightarrow \widehat{\delta}$$

$$E_{coup} \rightarrow \widehat{H}_J = - E_J \cos \widehat{\delta}$$

where $\left[\widehat{n}, \widehat{\delta}\right] = i$. The complete Hamiltonian of the circuit becomes

$$\widehat{H} = E_C \left(\widehat{n} - n_g\right)^2 - E_J \cos \widehat{\delta} \tag{6}$$

which reduces to [1]

$$\widehat{H}_{qubit} = \frac{\Delta}{2}\widehat{\sigma}_x + \frac{\epsilon}{2}\widehat{\sigma}_z \tag{7}$$

with

$$\Delta = -E_J$$
$$\epsilon = E_C\left(n_g - n - \frac{1}{2}\right)^2$$

Let us look and see what this perturbation does to the energy levels. To do this, we need to carry out a first-order stationary perturbation calculation. The first step is to build the perturbation matrix. The elements of this matrix are given by

$$\langle n'| E_J \cos\varphi |n\rangle . \quad (8)$$

We can explicitly calculate these values using the relation

$$|n\rangle = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\delta e^{-in\delta}|\delta\rangle$$

which gives us

$$\langle n'| E_J \cos\varphi |n\rangle = \frac{E_J}{2\pi}\int_0^{2\pi} d\delta e^{i\delta(n-n')}\cos\delta$$

There are only two interesting cases for this SC qubit, namely for $n' = n$ and $n' = n + 1$. In the first case we get

$$\frac{E_J}{2\pi}\int_0^{2\pi} d\delta e^{i\delta(0)}\cos\delta = 0$$

These are just the diagonal elements of the perturbation matrix. The ani-diagonal elements are obtained from the calculation for $n' = n+1$. This results in

$$\frac{E_J}{2\pi}\int_0^{2\pi} d\delta e^{-i\delta}\cos\delta = \frac{E_J}{2}$$

As a result, the perturbation matrix is

$$\widehat{H}_J = \frac{1}{2}\begin{pmatrix} 0 & E_J \\ E_J & 0 \end{pmatrix}$$
$$= \frac{E_J}{2}\widehat{\sigma}_x$$

This causes an anti-crossing at the degeneracy points. Diagonalizing this perturbation matrix gives us the energy splitting, and the new energy eigenstates of the perturbed qubit. The splitting, as shown in figure 6 on the following page, is equal to E_J, and the new eigenstates are the symmetric and anti-symmetric combinations of the unperturbed eigenstates:

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle - |n+1\rangle\right)$$
$$|1\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + |n+1\rangle\right)$$

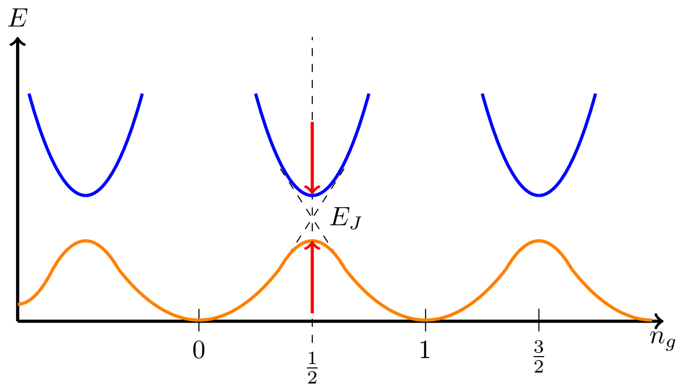

Figure 6: Plot of the eigenfunctions of the complete Hamiltonian. The perturbation causes anti-crossing at the degeneracy points, leading to two distinct energy levels.

As stated earlier, equation (6) on page 8 reduces to a Hamiltonian similar to that of a spin system,equation (7) on page 8. With this new Hamiltonian, we focus in on just one degeneracy point, and the energy levels look like those pictured in figure 7 on the next page.

Moving away from this "sweet spot" causes the energy splitting between states to change. Again, this can be calculated by diagonalizing the complete Hamiltonian of the qubit, equation (7) on page 8. The eigenvalues are

$$\lambda_{1,2} = \pm\frac{1}{2}\sqrt{\Delta^2 + \epsilon^2}$$

which means the energy splitting is $\nu = \sqrt{\Delta^2 + \epsilon^2}$.

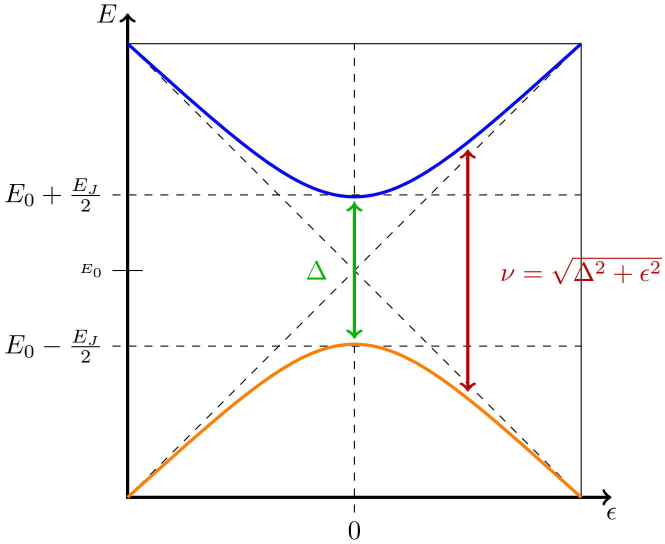

Figure 7: The "sweet spot" for the Cooper pair box and rf-SQUID.

3.2 The rf-SQUID

We can build upon the framework of the Cooper pair box just discussed to describe the rf-SQUID, or flux qubit. Figure 8 show the basic setup of an rf-SQUID, which in its simplest manifestation is a JJ with the SC ends shunted by a SC loop.

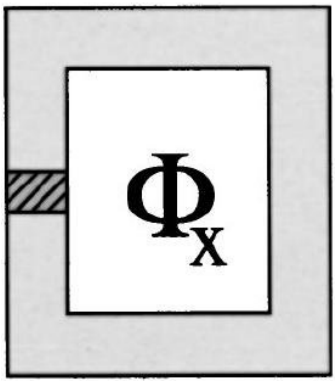

Figure 8: The flux qubit, or rf-SQUID [4].

The circuit is controlled through the use of an external magnetic flux, usually generated by a neighboring coil. The flux through the coil also changes the phase difference, δ, across the junction, due to the relation [4]

$$\frac{\delta}{2\pi} = \frac{\phi - \Phi_{\text{ext}}}{\Phi_0}$$

This and the fact that the coil has a self-induced flux through it changes the energy terms, and thus the Hamiltonian, from that seen in the Cooper pair box. We end up with [4]

$$\widehat{H} = \frac{2e^2}{C_J}\widehat{n}^2 + \frac{\widehat{\phi}^2}{2L_J} - E_J \cos\left(\frac{\widehat{\phi} - \Phi_{\text{ext}}}{\Phi_0}\right) \tag{9}$$

where $\left[\widehat{n}, \widehat{\phi}\right] = i\hbar$ By choosing large values of E_J and setting Φ_{ext} around $\Phi_0/2$, the last two terms of the Hamiltonian form a double well potential. Finally, by operating this circuit at low temperatures, one is able to is able to isolate the two lowest energy levels, effectively producing a qubit. The Hamiltonian of the circuit then reduces to a Hamiltonian similar to equation (7) on page 8. For the rf-SQUID, however, ϵ is the bias, or asymmetry of the double well, and for values of [4]

$$\beta_L - 1 = E_J / \left(\Phi_0/4\pi^2 L\right) - 1 \ll 1$$

is given by

$$\epsilon = -4\pi\sqrt{6\,(\beta_L - 1)}E_J\,(\Phi_{\text{ext}}/\Phi_0 - 1/2)\,.$$

The term Δ represents the tunneling amplitude between the two different well of the potential, which is also dependent of the barrier height between them and thus dependent on E_J.

4 Conclusion

This short introduction to SC qubits covered some of the basics of SC qubits. First it was shown that SC materials can be used to create integrated circuits, which exhibit quantum mechanical behavior "macroscopically". It is important for the implementation of these circuits as true qubits that, as already said, the degrees of freedom behave quantum mechanically, and that they are non-linear. The non-linearity ensures the isolation of two discrete energy levels.

There is one specific element that fulfills the two requirements named above, and that is the Josephson Junction. The JJ can be characterized as an inductance and capacitance connected in parallel. The behavior of this junction is fully described with two equations, the Josephson equations.

The JJs can be integrated into larger circuit designs to create different types of SC qubits. Two examples discussed in this paper were charge qubits and flux qubits.

The charge qubit consists of a JJ connected to a gate capacitance in series with a gate voltage. With this voltage source, control of the qubit is possible.

Finally, the flux qubit is just a JJ with both metallic ends shunted by a SC wire to create a loop. This qubit is controlled through the generation of an external magnetic field.

5 References

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