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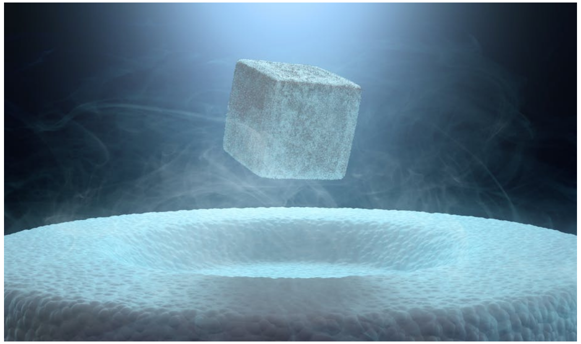
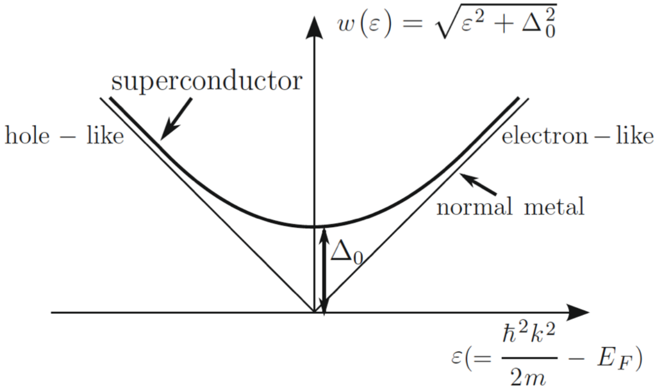
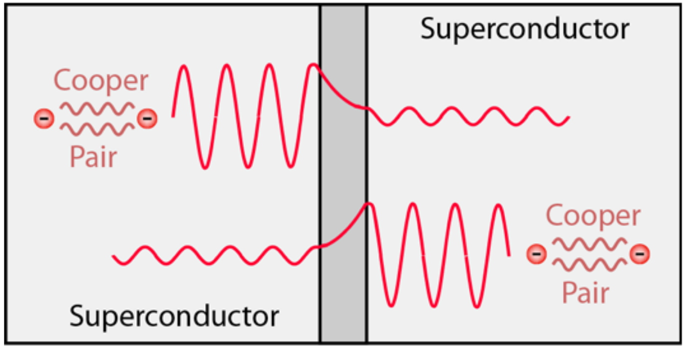
# Theory seminar: Superconducting quantum hardware for quantum computing

Universität Konstanz

# Lecture 1: BSC theory of superconductivity and physics of Josephson junctions



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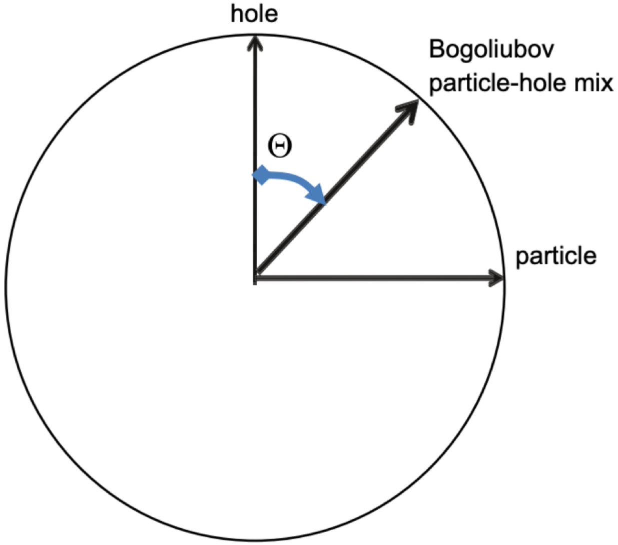
# Content:



1. The Cooper pair idea
2. The BCS ground state
3. Bogoliubov approach to BSC
4. Josephson junctions

*"Fermi surface is unstable with respect to a pair formation"*

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# 1. The Cooper pair idea

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- Consider the ground state of a free-electron gas at $T = 0$

  all electronic states are filled, up to the Fermi energy

  we now add 2 extra electrons to this system, and further assume

  a) the two electrons interact with each other via an attractive two-body potential $V(r_1, r_2)$
  b) Pauli exclusion principle (i.e. cannot occupy the states of the filled Fermi sea.)

- What is the lowest energy that this two-electron system can have?

$$\left\{ -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2}\right) + V(r_1, r_2) \right\} \Psi = E \Psi \qquad \Psi = \psi_{or}(r_1, r_2)\, \psi_s(\sigma_1, \sigma_2)$$

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As we are interested in the lowest possible energy, we consider a state with zero momentum

$$\Psi_{or}(r_1, r_2) = \sum_K g_K e^{iK\cdot r_1} e^{-iK\cdot r_2} = \sum g_K \left\{ \cos(K\cdot(r_1 - r_2)) + i \sin(K\cdot(r_1 - r_2)) \right\}$$

The spin part of the wavefunction can be either a singlet or a triplet state

$$\Psi_s(\sigma_1, \sigma_2) = \begin{cases} \text{Singlet } S=0 \\ \text{Triplet } S=1 \end{cases}$$

$$\Psi_{S=0}(\sigma_1, \sigma_2) = \frac{1}{\sqrt{2}}\left[ |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right]$$

$$\Psi_{S=1}(\sigma_1, \sigma_2) = \begin{cases} |\uparrow\rangle_1 |\uparrow\rangle_2 \\ \frac{1}{\sqrt{2}}\left[ |\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \right] \\ |\downarrow\rangle_1 |\downarrow\rangle_2 \end{cases}$$

**Question:** which spin state is consistent with the attractive-potential assumption?

The *singlet* spin state gives the larger probability amplitude for the electrons to be near each other.

$$\Psi = \sum_{k>k_f} g_k \cos(k(r_1-r_2))\, \Psi_{S=0}(\sigma_1,\sigma_2)$$

$$\Big\{ \underbrace{-\frac{\hbar^2}{2m}\Big(\frac{\partial^2}{\partial r_1^2}+\frac{\partial^2}{\partial r_2^2}\Big)}_{2\frac{\hbar^2 k^2}{2m} \equiv 2\epsilon_k} + V(r_1-r_2)\Big\}\Psi = E\Psi$$

$$V(r_1-r_2)\Psi = (E-2\epsilon_k)\Psi \qquad \text{(multiply by } \cos(k'(r_1-r_2))\, \Psi^*_{S=0}(\sigma_1,\sigma_2) \text{ and integrate)}$$

$$\Rightarrow (E-2\epsilon_{k'})\, g_{k'} = \sum_k V_{kk'}\, g_k \qquad V_{kk'} = \Omega^{-1}\int \cos(k'(r_1-r_2))\, V(r_1-r_2)\, \cos(k(r_1-r_2))\, d(r_1-r_2)$$

# 1. The Cooper pair idea

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assumption: $\begin{cases} V_{kk'} = -V & \hbar k, \hbar k' < \hbar\omega_D \\ V_{kk'} = 0 & \text{otherwise} \end{cases}$ $\rightarrow$ $(E - 2\epsilon_{k'})\, g_{k'} = -V \sum_k g_k$

$$\rightarrow g_{k'} = \frac{-V}{(E - 2\epsilon_{k'})} \sum_k g_k \rightarrow \underline{\sum_{k'} g_{k'}} = \sum_{k'} \frac{-V}{(E-2\epsilon_{k'})} \underline{\sum_k g_k}$$

$$\rightarrow \frac{1}{V} = \sum_{k' > k_f} \frac{1}{2\epsilon_{k'} - E} \equiv N(0) \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} \frac{d\epsilon}{2\epsilon - E} = \frac{1}{2} N(0) \ln \frac{2\epsilon_F - E + 2\hbar\omega_D}{2\epsilon_F - E}$$

# 1. The Cooper pair idea

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$$\frac{2}{N(0)V} = \ln\left[1 + \frac{2\hbar\omega_D}{2\epsilon_F - E}\right] \simeq \ln\left[\frac{2\hbar\omega_D}{2\epsilon_F - E}\right] \longrightarrow \frac{2\hbar\omega_D}{2\epsilon_F - E} = e^{\frac{2}{N(0)V}}$$

we use the *weak-coupling approximation:* $N(0)V \ll 1$

$$\boxed{E = 2\epsilon_F - 2\hbar\omega_D\, e^{-\frac{2}{N(0)V}}} < 2\epsilon_F$$

- The Fermi surface is *unstable* with respect to a pair formation if there is *an attractive interaction*.
- One electron polarizes the medium by attracting positive ions; these positive ions, in turn, attract the second electron.
- The above result cannot be obtained from perturbation theory.

## 2. The BCS ground state



The whole ground state is described by a *single* quantum phase

$$|\Psi_G\rangle = \prod_{k=k_1,\dots,k_m} \left( |u_k| + |v_k| e^{i\varphi} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right) |\phi_0\rangle$$

$$H_{BCS} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kl} V c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-l\downarrow} c_{l\uparrow} \qquad \Delta = -\sum_l V \langle c_{-l\downarrow} c_{l\uparrow} \rangle$$

$$H^{MF}_{BCS} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} - \sum_k \left( \Delta c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \Delta^* c_{-k\downarrow} c_{k\uparrow} \right) - \frac{|\Delta|^2}{V}$$

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$$c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} = a_k + \left( c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - a_k \right), \quad a_k = \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \rangle$$

$$c_{-l\downarrow} c_{l\uparrow} = b_l + \left( c_{-l\downarrow} c_{l\uparrow} - b_l \right), \quad b_l = \langle c_{-l\downarrow} c_{l\uparrow} \rangle$$

$$H_{AB} = AB$$

$$\rightarrow H^{MF}_{AB} = A\langle B\rangle + \langle A\rangle B - \langle A\rangle\langle B\rangle$$

# 3. Bogoliubov approach to BSC

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$$H^{MF}_{BCS} = \sum_{k\sigma} \epsilon_k c^{+}_{k\sigma} c_{k\sigma} - \sum_{k} \left( \Delta c^{+}_{k\uparrow} c^{+}_{-k\downarrow} + \Delta^{*} c_{-k\downarrow} c_{k\uparrow} \right) - \frac{|\Delta|^2}{V}$$

$$\boxed{\text{Bogoliubov transformation:} \quad \begin{aligned} c_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma^{\dagger}_{-\mathbf{k}\downarrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma^{\dagger}_{-\mathbf{k}\downarrow} - v_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} \end{aligned} \qquad \begin{aligned} c^{\dagger}_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} \gamma^{\dagger}_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow} \\ c_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow} - v_{\mathbf{k}} \gamma^{\dagger}_{\mathbf{k}\uparrow} \end{aligned}}$$

$$\begin{aligned} c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow} = & \left(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2\right) \left[\gamma^{\dagger}_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + \gamma^{\dagger}_{-\mathbf{k}\downarrow} \gamma_{-\mathbf{k}\downarrow}\right] \\ & + 2u_{\mathbf{k}} v_{\mathbf{k}} \left[\gamma^{\dagger}_{\mathbf{k}\uparrow} \gamma^{\dagger}_{-\mathbf{k}\downarrow} + \gamma_{-\mathbf{k}\downarrow} \gamma_{\mathbf{k}\uparrow}\right] + 2v_{\mathbf{k}}^2 . \end{aligned}$$

$$\begin{aligned} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = & -2u_{\mathbf{k}} v_{\mathbf{k}} \left[\gamma^{\dagger}_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + \gamma^{\dagger}_{-\mathbf{k}\downarrow} \gamma_{-\mathbf{k}\downarrow}\right] \\ & + \left(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2\right) \left[\gamma^{\dagger}_{\mathbf{k}\uparrow} \gamma^{\dagger}_{-\mathbf{k}\downarrow} + \gamma_{-\mathbf{k}\downarrow} \gamma_{\mathbf{k}\uparrow}\right] + 2u_{\mathbf{k}} v_{\mathbf{k}} . \end{aligned}$$

# 3. Bogoliubov approach to BSC



$$H^{MF}_{BCS} = \sum_{k\sigma} \epsilon_k \, c^\dagger_{k\sigma} c_{k\sigma} - \sum_k \left( \Delta \, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \Delta^* c_{-k\downarrow} c_{k\uparrow} \right) - \frac{|\Delta|^2}{V}$$

$$H^{MF}_{BCS} = \sum_k \left[ \epsilon_k (u_k^2 - v_k^2) + 2\Delta u_k v_k \right] \left[ \gamma^\dagger_{k\uparrow} \gamma_{k\uparrow} + \gamma^\dagger_{-k\downarrow} \gamma_{-k\downarrow} \right]$$

$$+ \sum_k \left[ \cancel{2\epsilon_k u_k v_k - \Delta (u_k^2 - v_k^2)} \right] \left[ \gamma^\dagger_{k\uparrow} \gamma^\dagger_{-k\downarrow} + \gamma_{-k\downarrow} \gamma_{k\uparrow} \right]$$

$$+ \sum_k \left[ 2\epsilon_k v_k^2 - 2\Delta u_k v_k \right] - \frac{\Delta^2}{V}$$

**Bogoliubov transformation**

$$c_{\mathbf{k}\uparrow} = u_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\downarrow}$$

$$c^\dagger_{-\mathbf{k}\downarrow} = u_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\downarrow} - v_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow}$$

$$c^\dagger_{\mathbf{k}\uparrow} = u_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow}$$

$$c_{-\mathbf{k}\downarrow} = u_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow} - v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\uparrow}$$

# 3. Bogoliubov approach to BSC

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$$|v_k|^2 = 1 - |u_k|^2 = \frac{1}{2}\left(1 - \frac{\epsilon_k}{E}\right); \quad E_k = \sqrt{\epsilon_k^2 + \Delta^2}$$

$$H_{BCS}^{MF} = \sum_k (\epsilon_k - E_k) - \frac{|\Delta^2|}{\lambda} + \sum_k E_k \left(\gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} + \gamma_{-k\downarrow}^\dagger \gamma_{-k\downarrow}\right)$$

- Ground state is the vacuum state for quasiparticles
- Quasiparticles are gapped from the ground state by the value given by Δ

– Superconducting gap Δ?

– Quasiparticle density of states?

# 3. Bogoliubov approach to BSC

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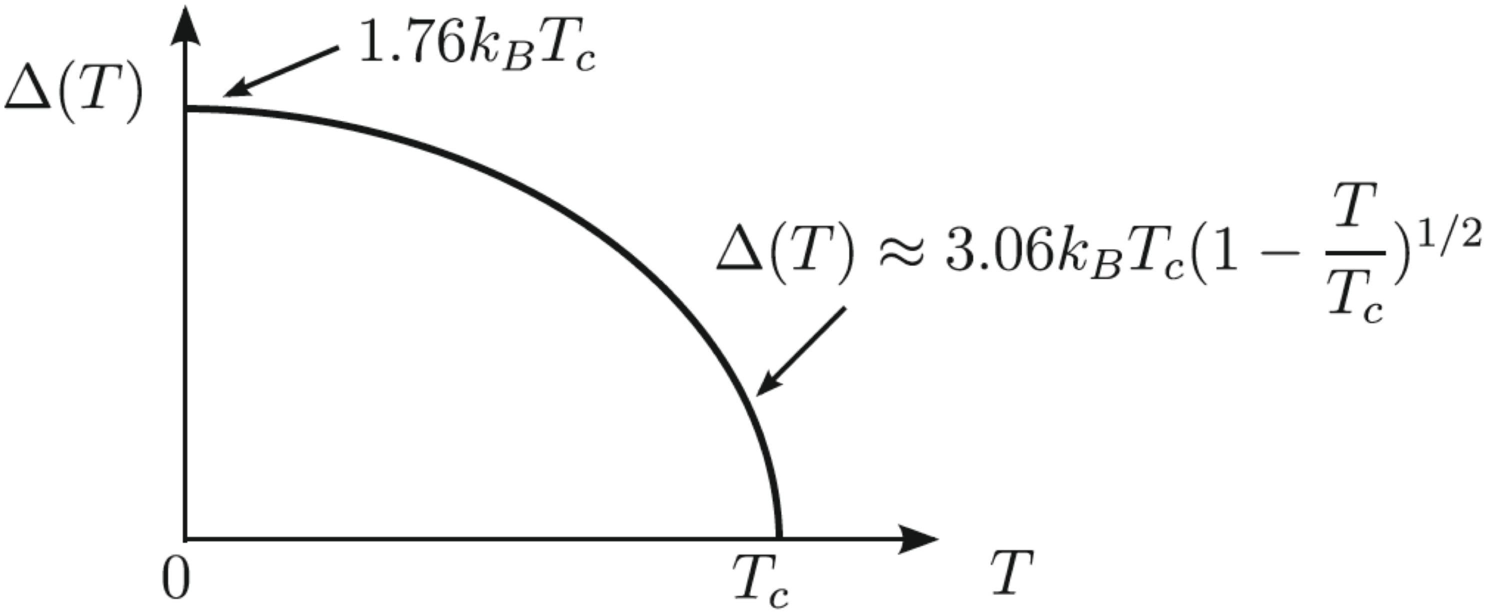
$$\Delta = -\sum_k V \langle c_{-k\downarrow} c_{k\uparrow} \rangle = -V \sum u_k^* v_k \langle 1 - \gamma^*_{k\uparrow} \gamma_{k\uparrow} - \gamma^+_{-k\downarrow} \gamma_{-k\downarrow} \rangle$$

$$\langle \gamma^+ \gamma \rangle \equiv f(E_k) : \text{Fermi-Dirac distribution} \qquad \langle 1 - \gamma^*_{k\uparrow} \gamma_{k\uparrow} - \gamma^+_{-k\downarrow} \gamma_{-k\downarrow} \rangle = 1 - 2f(E_k) = \tanh\left(\frac{E_k}{2k_B T}\right)$$

$$= \frac{1}{1 + e^{\frac{E_k}{k_B T}}}$$

$$\Delta = -V \sum \frac{\Delta}{2E_k} \tanh\left(\frac{E_k}{2k_B T}\right) = -N(0) V \int_0^{\omega_D} \frac{\Delta}{E_k} \tanh\left(\frac{E_k}{2k_B T}\right) dE_k \qquad E_k = \sqrt{\epsilon_k^2 + \Delta^2}$$

$$\boxed{1 = -N(0) V \int_0^{\omega_D} \frac{1}{\sqrt{\epsilon^2 + \Delta^2}} \tanh\left(\frac{\sqrt{\epsilon^2 + \Delta^2}}{2k_B T}\right) dE}$$




$$\boxed{1 = -N(0)V \int_0^{\omega_D} \frac{1}{\sqrt{\epsilon^2+\Delta^2}} \tanh\left(\frac{\sqrt{\epsilon^2+\Delta^2}}{2K_BT}\right) d\epsilon}$$

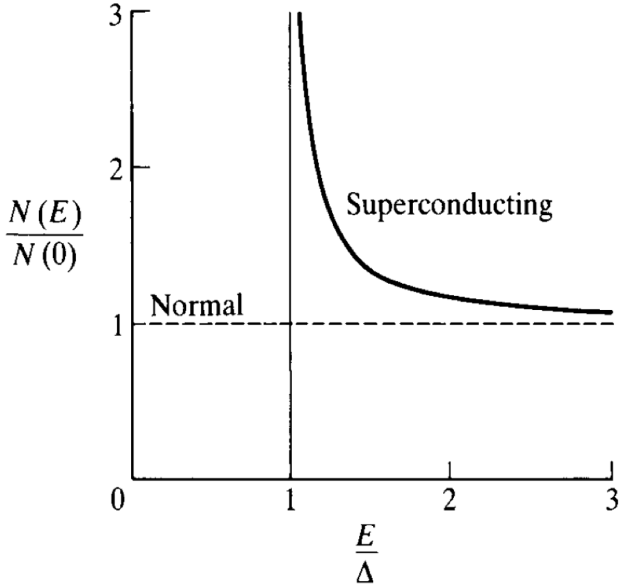
## 3. Bogoliubov approach to BSC



$$N_{qp}(E)\,dE = N_n(\epsilon)\,d\epsilon \simeq N_n(0)\,d\epsilon$$

$$E^2 = \epsilon^2 + \Delta^2 \rightarrow E\,dE = \epsilon\,d\epsilon$$

$$\rightarrow N_{qp}(E) = N_n(0)\frac{d\epsilon}{dE} = N_n(0)\frac{E}{\sqrt{E^2-\Delta^2}}$$




$$x_{qp} = \frac{\#\text{ of QPs}}{\#\text{ of Cooper pairs}}$$

$$x_{qp} = \frac{2}{N(0)\Delta}\int^{\infty} N_s(E)\,f(E)\,dE = \sqrt{\frac{2\pi KT}{\Delta}}\,e^{-\frac{\Delta}{T}}$$

$T = 20\text{ mK}$
$\Delta_{\text{Al}} \equiv 2.1\text{ K}$ $\longrightarrow$ $x_{\text{qp}}^{eq} \propto 10^{-47}$

However, values deduced from lots of experiments indicate $x_{qp} = [10^{-9}, 10^{-6}]$

## 4. Josephson junctions

- Josephson junction are building blocks of superconducting qubits

- a supercurrent flows through the device even in absence of an external bias due to the phase difference.

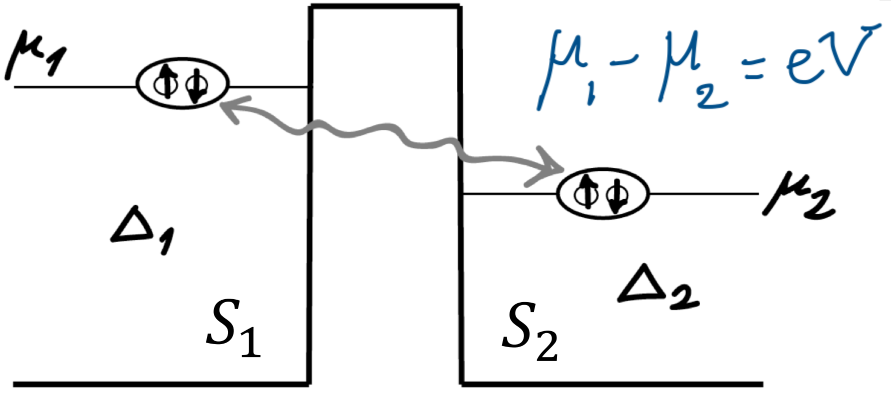



Fig. 2.2. Josephson effects in a tunnelling junction.

$$\psi_1 = \sqrt{\frac{n_{s1}}{2}}\, e^{i\varphi_1} \qquad \psi_2 = \sqrt{\frac{n_{s2}}{2}}\, e^{i\varphi_2} \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad H = \begin{pmatrix} eV & K \\ K & -eV \end{pmatrix}$$

$$-i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\frac{n_{s1}}{2}}\,e^{i\varphi_1}\\ \sqrt{\frac{n_{s2}}{2}}\,e^{i\varphi_2}\end{pmatrix}=\begin{pmatrix}eV & K\\ K & -eV\end{pmatrix}\begin{pmatrix}\sqrt{\frac{n_{s1}}{2}}\,e^{i\varphi_1}\\ \sqrt{\frac{n_{s2}}{2}}\,e^{i\varphi_2}\end{pmatrix}$$

$$-i\hbar\left[\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n_{s1}}}\frac{d}{dt}n_{s1}+i\sqrt{\frac{n_{s1}}{2}}\frac{d\varphi_1}{dt}\right]e^{i\varphi_1}=eV\sqrt{\frac{n_{s1}}{2}}\,e^{i\varphi_1}+K\sqrt{\frac{n_{s2}}{2}}\,e^{i\varphi_2}$$

$$\rightarrow\quad \frac{d}{dt}n_{s1}+i2n_{s1}\frac{d\varphi_1}{dt}=\frac{i}{\hbar}\left\{2eVn_{s1}+2K\sqrt{n_{s1}n_{s2}}\,e^{i(\varphi_2-\varphi_1)}\right\}$$

$$-i\hbar\left[\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n_{s2}}}\frac{d}{dt}n_{s2}+i\sqrt{\frac{n_{s2}}{2}}\frac{d\varphi_2}{dt}\right]e^{i\varphi_2}=K\sqrt{\frac{n_{s1}}{2}}\,e^{i\varphi_1}-eV\sqrt{\frac{n_{s2}}{2}}\,e^{i\varphi_2}$$

$$\rightarrow\quad \frac{d}{dt}n_{s2}+i2n_{s2}\frac{d\varphi_2}{dt}=\frac{i}{\hbar}\left\{2K\sqrt{n_{s1}n_{s2}}\,e^{-i(\varphi_2-\varphi_1)}-2eVn_{s2}\right\}$$

$$\rightarrow\quad \frac{d}{dt}n_{s1}=\frac{-1}{\hbar}2K\sqrt{n_{s1}n_{s2}}\,\sin(\varphi_2-\varphi_1)=-\frac{d}{dt}n_{s2}$$

$$\frac{d}{dt} n_{s1} = \frac{-1}{\hbar} 2K \sqrt{n_{s1} n_{s2}} \sin(\varphi_2 - \varphi_1) = -\frac{d}{dt} n_{s2}$$

$$\rightarrow \quad I_s = I_c \sin(\varphi_1 - \varphi_2)$$

$$2 n_{s1} \frac{d\varphi_1}{dt} = \frac{1}{\hbar} \left\{ 2eV n_{s1} + 2K \sqrt{n_{s1} n_{s2}} \cos(\varphi_2 - \varphi_1) \right\}$$

$$2 n_{s2} \frac{d\varphi_2}{dt} = \frac{1}{\hbar} \left\{ 2K \sqrt{n_{s1} n_{s2}} \cos(\varphi_2 - \varphi_1) - 2eV n_{s2} \right\}$$

$$\frac{d\varphi_1}{dt} - \frac{d\varphi_2}{dt} = \frac{1}{\hbar} 2eV + \cos(\varphi_2 - \varphi_1) 2K \left[ \sqrt{\frac{n_{s2}}{n_{s1}}} - \sqrt{\frac{n_{s1}}{n_{s2}}} \right]$$

$$\rightarrow \quad V = \frac{\hbar}{2e} \cdot \frac{\mathrm{d}}{\mathrm{d}t} (\phi_2 - \phi_1).$$