Universität
Konstanz

Integrated Physics Course IV
Exp.-Section – Atomic Physics
SoSe 19

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Problem Set 3

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Exercise 6: Hermitian operator (I) **(1 Cross)**

a) What condition does an operator have to fulfill to be a self-adjoint operator (Hermitian operator)?

b) Now insert the momentum operator!

c) Choose a side of your equation and rearrange it through partial integration ($\int_a^b udv = uv|_a^b - \int_a^b vdu$).

d) The term $uv|_a^b$ must disappear due to a general property of wavefunctions - which one?

e) Complete the proof that the momentum operator is Hermitian!

Exercise 7: Hermitian operator (II) **(1 Cross each for a-b, c-d)**

The definition of angular momentum in quantum mechanics is similar to classical mechanics $\vec{L} = \vec{r} \times \vec{p}$ with only the difference that the individuals variables x, y, z and p_x, p_y, p_z are replaced by the corresponding operators $\hat{x}, \hat{p}_x$....

a) Is $\vec{L}$ an operator? *(Note: use the formal definition of an operator.)*

b) Check if the components of $\vec{L}$ are Hermitian. Do not calculate explicitly with the help of a test function and an integral, but use the following facts: $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$, $[\hat{x}, \hat{p}_y] = [\hat{x}, \hat{p}_z] = 0$ etc. $\hat{x}, \hat{p}_x$... are Hermitians. Furthermore $(\hat{S}\hat{T})^\dagger = \hat{T}^\dagger \hat{S}^\dagger$ for any operators $\hat{S}$ and $\hat{T}$.

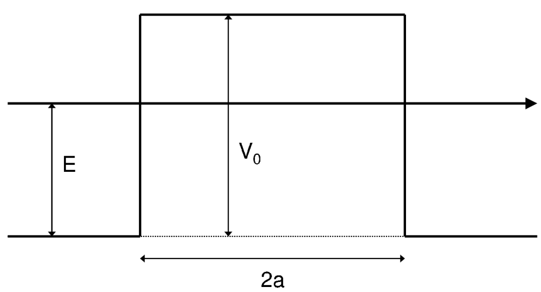
c) In preparation for the next problem, calculate the following commutators: $[\hat{L}_x, \hat{z}]$, $[\hat{L}_x, \hat{p}_z]$, $[\hat{L}_x, \hat{x}]$ and $[\hat{L}_x, \hat{p}_x]$. use known commutators instead of calculating them explicitly!

d) Now proof that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. Cyclic permutations of this relationship are also true (no need to proof). Can the total angular momentum $\vec{L}$ be determined with arbitrary accuracy?

Exercise 8: $\alpha-$decay and tunnel effect (written exercise) **(6 points)** The transmission probability through a symmetrical rectangular barrier of width $2a$ is defined as follows:

$$|T|^2 = \frac{1}{1 + (1 + (\varepsilon^2/4))\sinh^2(2\kappa a)}$$

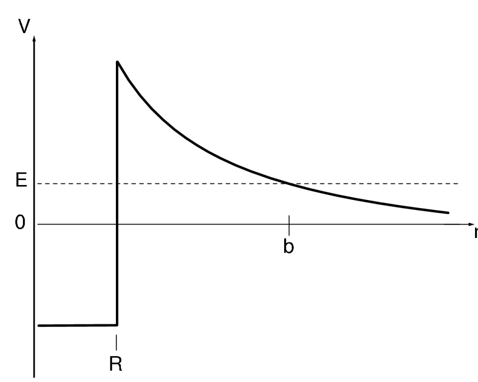
where $\varepsilon = \frac{\kappa}{k} - \frac{k}{\kappa}$, $k = \frac{\sqrt{2mE}}{\hbar}$, $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$.



a) Proof that for a very thick barrier i.e. $\kappa a \gg 1$ the probability of transmission can be well approached as:

$$|T|^2 \approx \frac{16E(V_0 - E)}{V_0^2} \exp(-4\sqrt{2m(V_0 - E)}\frac{a}{\hbar}) \qquad (*)$$

b) As an example of tunneling through a potential barrier we want to consider at the $\alpha-$decay, that is, the emission of a particle (He-core = 2 protons and 2 neutrons) from an atomic nucleus.
For the particle, the potential caused by the remaining core constituents roughly looks like this: Within nuclear radius R, it is bound by nuclear forces (deep potential well). If the $\alpha-$particle moves away from R beyond the center of the nucleus, $\alpha-$particles and remnant nuclei behave like two corresponding positive point charges. For $r > R$ we use the Coulomb potential. What is $V(r)$ for $r > R$



If the particle already has an energy $E > 0$ in the nucleus, it can tunnel through the barrier and leave the nucleus. The barrier is not rectangular, and the potential level outside is not the same on both sides either. For a given E, one can first determine the location (radius) b, where the exit from the barrier occurs, and thus the barrier thickness. To apply (*), simplify the potential as sketched below. Use $V = 0$ for $r < R$ and $r > b$. Take the mean of all values of the $1/r-$ potential between R and b and thus determine a V_0 as the height of a rectangular barrier, where the "sloped" barrier can be reasonably replaced. Polonium is converted by $\alpha-$decay into lead (corresponding isotopes, where the neutron number is preserved after loosing the $\alpha-$particle). Keep using $R = 1 \cdot 10^{-14}$m and $E = 10$ MeV as given numerical values. Calculate b, V_0 and $|T|^2$.