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3. **The Spherical Model**

The n-vector model (often denoted the $O(n)$ model) is a useful model in statistical physics in which n-component classical spins of fixed length are placed on the vertices of a lattice of dimension d. The Hamiltonian for this model is given by

$$H = \frac{1}{2} \sum_{i,j} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j ,$$

where J_{ij} is the coupling between sites i and j. The spin variable $\mathbf{s}_i$ is an n-component vector $\mathbf{s}_i = (s_i^{(1)}, s_i^{(2)}, \cdots, s_i^{(n)})$, where i labels the lattice site and there are N sites in total. The vector $\mathbf{s}_i$ is subject to the constraint that $\mathbf{s}_i \cdot \mathbf{s}_i = n$. Special cases of the model are $n = 0$ (self avoiding walk), $n = 1$ (Ising model), $n = 2$ (XY model) and $n = 3$ (Heisenberg model). In 1968 H.E. Stanley showed that the $n \to \infty$ limit of the n-vector model is equilivilent to the Berlin-Kac spherical model, first introduced in 1952. The advantage of studying the spherical model is that it is exactly soluble and yields non-classical values for the critical exponents.

(a) Why is the integral representation

$$Z(K) = \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_N \, W(\{s_N\}) \exp\left(-\frac{1}{2} \beta \sum_{i,j} J_{ij} s_i s_j \right) , \qquad (1)$$

equivilent to the standard expression for the partition function of the Ising model when the weight function is given by $W(\{s_N\}) = \prod_{i=1}^{N} \delta(s_i^2 - 1)$?

(b) The spherical model is a generalization of the Ising model in which the spin variables are allowed to take a continuous range of values ($-\infty < s_i < \infty$). The spherical model partition function is again given by Eq.(1) but with the weight function $W(\{s_N\}) = \delta\left(\sum_{i=1}^{N} s_i^2 - N \right)$. Discuss the differences between the spherical and Ising models. Why do we call this model 'spherical'?

(c) The delta function can be usefully expressed using the Laplace representation

$$\delta\left(\sum_i s_i^2 - N \right) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp' \exp\left(p' \left(N - \sum_i s_i^2 \right) \right) .$$

Use the identity $-\frac{1}{2}\beta \sum_{i,j} J_{ij} s_i s_j = N\alpha - \alpha \sum_i s_i^2 - \frac{1}{2}\beta \sum_{i,j} J_{ij} s_i s_j$, to show that the partition function is given by

$$Z = \frac{e^{N\alpha}}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} dp \, e^{pN} \int ds_1 \cdots \int ds_N \exp\left(-\sum_{ij} \left(p\delta_{ij} + \frac{1}{2}\beta J_{ij} \right) s_i s_j \right) , \qquad (2)$$

where $p \equiv p' + \alpha$ for arbitrary α. Why was it necessary to introduce the parameter α? (*HINT: Consider the convergence of the integrals*).

(d) Assume translational invariance $J_{ij} = J_{i-j}$ to show that

$$Z = \frac{\pi^{N/2} e^{N\alpha}}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} dp \exp\left(pN - \frac{1}{2}\sum_{\mathbf{q}} \log\left(p + \frac{1}{2}\beta J_{\mathbf{q}}\right)\right), \tag{3}$$

where $J_{\mathbf{q}} \equiv \sum_{\mathbf{j}} J_{\mathbf{j}}\, e^{-2\pi i(\mathbf{j}\cdot\mathbf{q})/L}$ is the discrete Fourier transform of J_{i-j}.

(e) We now specify to nearest neighbour interactions for which $J_{ij} = -\epsilon$ (i, j nearest neighbours) and $J_{ij} = 0$ (otherwise). First show that $J_{\mathbf{q}} = -2\epsilon \sum_{l=1}^{d} \cos(2\pi q_l/L)$, where $L \equiv N^{1/d}$. Next, replace the sum by an integral to show that

$$Z = (\beta\epsilon)^{1-N/2} \frac{\pi^{N/2} e^{N\alpha}}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} d\xi e^{\,g(\xi)}, \tag{4}$$

where $\xi \equiv p/\beta\epsilon$ and α' is a large real number. The function $g(\xi) \equiv N(\beta\epsilon\xi - \phi(\xi)/2)$, where

$$\phi(\xi) = \frac{1}{(2\pi)^d} \int_0^{2\pi} d\omega_1 \cdots \int_0^{2\pi} d\omega_d \log\left(\xi - \sum_{k=1}^{d} \cos(\omega_k)\right)$$

and $\omega_l \equiv 2\pi q_l/L$.

(f) Following all this rearrangement, the result (4) is suitable for approximation by the method of steepest descents, which becomes exact in the limit $N \to \infty$. Use this approximation method to show that

$$Z \approx (\beta\epsilon)^{1-N/2} \pi^{N/2} e^{N\alpha} \frac{e^{g(\xi_s)}}{\sqrt{2\pi g''(\xi_s)}}, \tag{5}$$

where ξ_s is the location of the maximum in $g(\xi)$, obtained from solution of the equation

$$2\beta\epsilon = \frac{1}{(2\pi)^d} \int_0^{2\pi} d\omega_1 \cdots \int_0^{2\pi} d\omega_d \frac{1}{\xi_s - \sum_k \cos(\omega_k)}. \tag{6}$$

In $d = 1$ and $d = 2$ the spherical model exhibits no phase transition. In $d = 3$ it can be shown that ξ_s is a smooth function of β only for $\beta < 0.25272/\epsilon$, thus identifying a critical point, $\beta_c = 0.25272/\epsilon$. Take the logarithm of Z followed by the limit $N \to \infty$ to obtain the exact free energy per site of the spherical model

$$\beta f = \frac{1}{2}\log(\beta\epsilon/\pi) - \beta\epsilon\xi_s + \frac{1}{2}\frac{1}{(2\pi)^d} \int_0^{2\pi} d\omega_1 \cdots \int_0^{2\pi} d\omega_d \log\left(\xi_s - \sum_{k=1}^{d} \cos(\omega_k)\right) - \alpha \tag{7}$$

(g) We can now use our results to calculate some critical exponents. Specializing to $d = 3$, prove that near β_c we have $(\xi_s - 3) \sim (\beta_c - \beta)^2$ (*HINT: The integral in (6) is dominated by the low ω behaviour of the integrand*). Thus calculate the susceptibility and show that $\chi \sim |t|^{-\gamma}$, where $t = (T - T_c)/T_c$, with exponent $\gamma = 2$. Calculate the internal energy per site $u = \frac{d\beta f}{d\beta}$ and thus the specific heat per site $c = -\beta^2 \frac{du}{d\beta}$. Show that the specific heat exponent $\alpha = -1$, where $c \sim |t|^{-\alpha}$), i.e. there is no specific heat anomaly. The remaining exponents of the spherical model can all be calculated exactly, but require more involved calculations. The spherical model values $\beta = \frac{1}{2}$, $\delta = 5$, $\eta = 0$ and $\nu = 1$ should be contrasted with the results from mean field theory and the Gaussian model.