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3. **The Gaussian Approximation**
In this problem we shall introduce the Gaussian approximation to incorporate fluctuation effects and will use this technique to calculate the critical exponents from a Landau free energy. The Gaussian approximation provides the lowest order systematic correction to mean field theory by assuming that the fluctuations are independent random variables. The Landau free energy is a functional of the order parameter field

$$L[\eta] = \int d^d\mathbf{r} \left(\frac{1}{2}\gamma(\nabla\eta(\mathbf{r}))^2 + at\eta^2(\mathbf{r}) + \frac{1}{2}b\eta^4(\mathbf{r}) \right) + a_0 V, \tag{1}$$

where γ, a, b (all positive) and a_0 are phenomenological parameters and $t = (T - T_c)/T_c$ is the temperature relative to the critical point.

(a) It is convenient to work in Fourier space using the transform pair

$$\eta_{\mathbf{k}} = \int d^d\mathbf{r}\, \eta(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}} \qquad \eta(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} \eta_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}.$$

Note that by retaining the discrete version of the back transform we will be able to perform functional integrals before taking the continuum limit. Set $b = 0$ in (1) and show that the Landau free energy can be expressed as

$$L[\eta] = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2}|\eta_{\mathbf{k}}|^2[2at + \gamma k^2] + a_0 V. \tag{2}$$

Mean field theory only considers states with $\mathbf{k} = 0$, corresponding to a spatially constant $\eta(\mathbf{r})$. States with $\mathbf{k} \neq 0$ represent fluctuations. Why is it necessary to introduce an upper limit Λ to the sum over $\mathbf{k}$?

(b) The free energy is related to the Landau function by a functional integration over all possible order parameter fields

$$Z = e^{-\beta F} = \int \mathcal{D}\eta\, e^{-\beta L[\eta]}. \tag{3}$$

In general $\eta_{\mathbf{k}}$ is complex and both real and imaginary parts can be varied independantly. The functional measure is then given by a product of integrals

$$\int \mathcal{D}\eta \equiv \int \prod_{|\mathbf{k}|<\Lambda} d(\operatorname{Re}\eta_{\mathbf{k}})d(\operatorname{Im}\eta_{\mathbf{k}}),$$

Evaluate the functional integral (3) using the Landau function (2) to obtain the free energy

$$F = a_0 V - \frac{1}{2} k_B T \sum_{|\mathbf{k}|<\Lambda} \log\left(\frac{2\pi V k_B T}{2at + \gamma k^2}\right). \tag{4}$$

As the field $\eta(\mathbf{r})$ is taken to be real we have introduced a factor of 1/2 in front of the summation. Why? (*HINT: consider the relation between* $\eta_{\mathbf{k}}$ *and* $\eta_{-\mathbf{k}}$ for real $\eta(\mathbf{r})$).

(c) The two-point correlation function can similarly be calculated using a functional integral.

$$\langle \eta_{\mathbf{k}} \eta_{\mathbf{k}'} \rangle = \frac{1}{Z} \int \mathcal{D}\eta \; \eta_{\mathbf{k}} \, \eta_{\mathbf{k}'} \, e^{-\beta L[\eta]}, \tag{5}$$

where Z is given by (3). Show that this correlation function is given by

$$\langle |\eta_{\mathbf{k}}|^2 \rangle = \frac{k_B T V}{2at + \gamma k^2} \equiv V G_{\mathbf{k}} \tag{6}$$

for $\mathbf{k} = -\mathbf{k}'$ and that it is zero otherwise. Use the asymptotic $k \to \infty$ result $G_{\mathbf{k}} \sim k^{-2+\eta}$ to find the value of the exponent η and the sum rule $k_B T \chi_T = G_{\mathbf{k}=0}$ to find the exponent γ, where $\chi_T \sim |t|^{-\gamma}$. The correlation function can be written in the form

$$G_{\mathbf{k}} = \frac{k_B T}{\gamma (k^2 + \xi^{-2})},$$

where ξ is the correlation length. Find the exponent ν for the divergence of ξ at the critical point ($\xi \sim |t|^{-\nu}$).

(d) Express the real space correlation function $G(\mathbf{r}, \mathbf{r}') = \langle \eta(\mathbf{r}) \eta(\mathbf{r}') \rangle$ as a sum over $\mathbf{k}$ vectors. How can the translational invariance of the system be identified from this result?

(e) We now calculate the heat capacity from (4). The heat capacity is given by

$$c = -T \frac{\partial^2 (F/V)}{\partial T^2}. \tag{7}$$

Perform the derivative to obtain $c = A + B$, where

$$A \equiv \frac{k_B T}{2 V T_c^2} \sum_{|\mathbf{k}|<\Lambda} \frac{4a^2}{(2at + \gamma k^2)^2} \qquad B \equiv -\frac{k_B}{V T_c} \sum_{|\mathbf{k}|<\Lambda} \frac{2a}{(2at + \gamma k^2)}.$$

We will look at the two terms A and B seperately. Replace the summation in the expression for A by an integral and make a change of variables $\mathbf{q} = \xi \mathbf{k}$, where ξ is the correlation length, in order to extract the divergent behaviour. By considering the behaviour of the integral in various dimensions d show that $A \propto \xi^{4-d} \sim t^{-(2-d/2)}$, for $d < 4$ and that it remains finite for $d > 4$. Repeat this procedure for B to show that the specific heat behaves as $C \sim t^{-(2-d/2)}$ for $d < 4$ and remains finite for $d > 4$. We have thus shown that the specific heat exponent $\alpha = 2 - d/2$ in the Gaussian approximation.

(f) In parts (a)-(e) we have considered states with $T > T_c$ by considering fluctuations about $\eta(\mathbf{r}) = 0$. For $T < T_c$ we need to expand about one of the two spontaneously occuring minima in the free energy. We thus replace the potential for fluctuations by a harmonic potential. Using $\eta(\mathbf{r}) = \eta_s + \psi(\mathbf{r})$, where $\eta_s = \pm(-at/b)^{1/2}$, calculate the Landau free energy to quadratic order in $\psi(\mathbf{r})$ and express your answer in the Fourier components of $\psi(\mathbf{r})$. (*HINT: Once you have obtained the Landau free energy in terms of* ψ *there is no need for a new calculation to arrive at the required result*).