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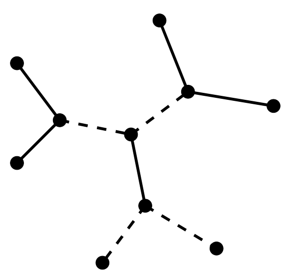
Renormierungsgruppe und Feldtheorie
Sommersemester 2008

Übungsblatt 2, Ausgabe 14.05.2008, abzugeben bis 21.05.2008

2. **Percolation**
Fluids do not pass through a solid with a small concentration of holes. However, beyond a certain threshold concentration, the holes overlap, and the fluid can percolate through a connected channel in the material. Percolation is a classical geometric phase transition. We will consider models for percolation theory based on lattices. The lattice bonds can either be occupied or empty with probabilities p and $(1-p)$ respectively. If two lattice sites which are (infinitely) far apart are connected by occupied bonds, the bonds are said to percolate the lattice.

(a) We will first consider the Bethe lattice. The site at the origin has z bonds to the next sites, which themselves have $z-1$ bonds to next sites and so on. The figure below shows the Bethe lattice for $z=3$. In this example, full lines correspond to occupied bonds, dashed lines to empty bonds.

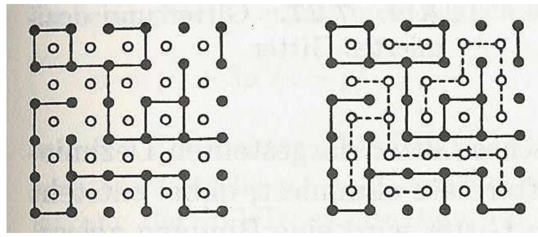


i. Consider an infinitely large Bethe lattice for $z=3$. Determine the probability Q, that the origin is *not* connected to infinity via *one* of its bonds.
hint: Q is given by the probability that the first bond is empty $(1-p)$ plus the probability that it is occupied (p) times the probability that the next site is not connected to infinity. The latter can be expressed in terms of Q in an infinite lattice which gives you a closed equation for Q.

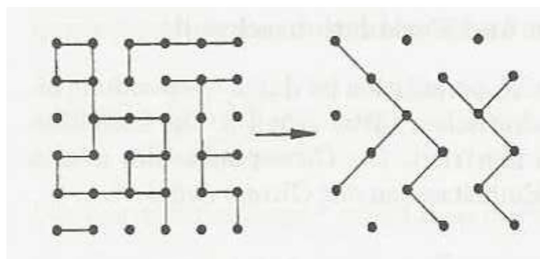
ii. Using your result for Q, determine the probability P_∞, that the origin *is* connected to infinity. Determine the percolation threshold p_c, where $P_\infty = 0$ holds for $p \le p_c$.

iii. Determine the critical exponent β defined by $P_\infty \propto |p_c - p|^\beta$ for $p_c \to p$ from above.

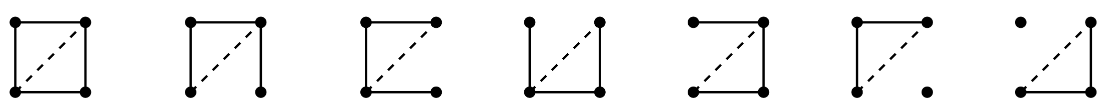
(b) Duality has a very natural interpretation in percolation. As an example, we consider the bond percolation problem on a square lattice ($d=2$). The sites of the dual lattice are shifted half the lattice constant in both directions. In the figure below, you see the lattice (full circles) and the dual lattice (open circles). Occupied bonds are drawn in full lines, occupied dual bonds in dashed lines. If a bond is occupied, its dual bond is empty and vice versa. What follows for the percolation of the dual bonds if the bonds percolate? What is the percolation threshold p_c?



(c) Now we want to consider the square lattice via renormalization group in position space. The procedure works as follows: In each renormalization step, we will remove every other site from the lattice, see the figure below. This leads to a square lattice which is tilted by 45°. Two sides in the reduced lattice are connected, if they were connected in the original lattice via two rectangular bonds (the shortest possible way). We then calculate the probability p', that two lattice sites in the reduces lattice are connected. After (infinitely) many renormalization steps, p' will be the percolation probability P_∞, equivalent to the Bethe lattice.



i. Find p' as function of p for the first renormalization step by summing up the probabilities for the configurations below. This is our RG recursion relation.



ii. Find the fixed points of this relation. There is one nontrivial fixed point. Is it stable or unstable? What follows for the value of the percolation threshold p_c? How does it compare to your result from b)?

iii. Approaching the threshold from below, the correlation length ξ of the bonds diverges as

$$\xi \propto |p - p_c|^{-\nu}.$$

Assuming that regions smaller than ξ are self similar upon renormalization, it follows that ξ' of the renormalized lattice is identical to ξ. Noting that ξ' is measured in rescaled units of length, we find

$$b|p' - p_c|^{-\nu} = |p - p_c|^{-\nu}.$$

What is b in our case? Solve this equation for ν for $|p - p_c| \ll 1$. Compare it to the exact value of $\nu = \frac{4}{3}$.