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Renormierungsgruppe und Feldtheorie
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1. (a) **Hubbard Stratonovich Transformation**
In this problem you will use one of the most general methods for turning statistical mechanics problems into field theories. Mean field theory will be introduced by taking the maximum term in the partition function.
The Hamiltonian for the Ising spin system with external Field H_i and nearest neighbor interaction is given by J_{ij}:

$$H\{S\} = -\frac{1}{2}\sum_{i\neq j} J_{ij} S_i S_j - \sum_i H_i S_i \tag{1}$$

where $J_{ij} = J > 0$ if i and j are nearest neighbors and $J_{ij} = 0$ otherwise.
In the following Einstein's sum convention will be used. (Repeated indices are to be summed over).

i. First prove the identity

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} x_i A_{i,j} x_j + x_i B_i\right) = \frac{1}{\sqrt{\det A}} \exp\left(\frac{1}{2} B_i (A^{-1})_{ij} B_j\right) \tag{2}$$

where A is a real, symmetric, positive definite matrix, an B is an arbitrary vector.
Hint: Use the transformation: $y_i = x_i - (A^{-1})_{ij} B_j$

ii. The identity above can be used to make the term in the Hamiltonian with $S_i S_j$ linear in S_i. Why can't this be done right away?
Hint: Redefine the zero point energy.

iii. Apply the identity of part (1()i), making the identification $A^{-1}_{ij} = J_{ij}$ and $B_i = S_i$. Show that

$$Z = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\psi_i \exp\left(-\beta S(\{\psi_i\}, \{H_i\}, \{J_{ij}\})\right) \tag{3}$$

where

$$S = \frac{1}{2}\left(\psi_i - H_i\right) J^{-1}_{ij} \left(\psi_j - H_j\right) - \frac{1}{\beta}\sum_i \ln(2\cosh\beta\psi_i) \tag{4}$$

The partition function is now a functional integral in the limit that the lattice spacing $a \to 0$ and $N \to \infty$. Then the dummy variable ψ_i is a function $\psi(\mathbf{r})$

iv. Assume that Z can be approximated by the maximum term in the functional integral: $Z \approx \exp(-\beta S(\bar{\psi}_i))$ where $\bar{\psi}_i$ is the value of the field ψ_i which minimizes S. Find the equation satisfied by $\bar{\psi}_i$, and show that the magnetization at site i

$$m_i \equiv \langle S_i \rangle = -\partial F / \partial H_i \approx \partial S / \partial H_i \tag{5}$$

is given by $m_i = \tanh \beta \bar{\psi}_i$. Hence find the equation of state: $H_j(\{m_i\})$.

v. Now consider the case of uniform magnetization $m_i = m$ on a $d-$dimensional hypercubic lattice, with coordination number $z = 2d$. This leads to an implicit equation for the magnetization m. Discuss (graphically) $H(m)$.
Hint: Consider $H = 0$ first and then the general case

vi. Find the critical point ($T_c = \frac{2dJ}{k_B}$), i.e. the temperature when spontaneous magnetization can occur. Make a Taylor expansion of the equation of state $H(m)$. Determine the critical exponents for the spontaneous magnetization $m_0 \propto (T_c - T)^\beta$, the critical isotherm $H \propto m^\delta$ and the isothermal magnetic susceptibility $\chi \propto |T - T_c|^{-\gamma}$

vii. Plot $m(H)$
Hint: In (1()v) the solution $m(H)$ for $T < T_c$ is not always unique. Explain which solution has to be taken!

viii. Plot $m(T)$.