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Übungen zu Zeitabhängige Phänomene
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Übungsblatt 6: How physics can make you rich

The Black-Scholes formula In 1973 F. Black and M. Scholes applied methods of stochastic calculus to predict the prices of stock options. This work had a huge impact on the field of economics and is now routinely used in financial markets around the world for the pricing of so-called European options. In 1997 Scholes recieved the Nobel prize in economics. The Black-Scholes formula follows from a Langevin equation and is a partial differential equation for the option price.
We consider a financial product (asset) with price $S(t)$ at time t. This is our basic stochastic variable, analogous to the position of a diffusing Brownian particle. A European option is an agreement between buyer and seller made at time t. In this agreement the seller agrees that at time $T > t$ he will sell the asset for a given fixed price K. However, the buyer can choose at time T whether he wants to buy or not. If the current market price $S(T) < K$ then the buyer will buy the asset from somewhere else. If $S(T) > K$ then the buyer will buy the asset from the agreed seller for price K. Of course, the seller doesn't make this agreement for free. At time t the seller charges the potential buyer an amount $C(S,T)$ for signing the agreement. $C(S,T)$ is called the price of the option and is the quantity we wish to calculate.

1. The asset price is approximately described by the Langevin equation

$$\frac{d}{dt}S(t) = \alpha S(t) + \sigma S(t) w(t)$$

where α is a constant drift value and $w(t)$ is a white noise, $\langle w \rangle = 0$, $\langle w(t)w(t')\rangle = 2\delta(t-t')$, with constant σ. Discuss the assumptions behind this equation and whether these seem reasonable for the present financial application. Derive the Kramers-Moyall coefficients and thus obtain the Fokker Planck equation for the distribution of the stochastic variable S. Make the change of variable $Y = \ln(S)$ to obtain simpler Langevin equation for $Y(t)$ and find the corresponding Fokker-Planck equation.

2. The price of the option is $C(S,t)$. Using the expression for the total derivative $\frac{dC}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\frac{dS}{dt}$ determine the Kramers-Moyall coefficients and Fokker-Planck equation for C. Hint: First identify the drift and diffusion terms in the Langevin equation for C.

3. A portfolio Π related to C and S by a Legendre transform $\Pi = C - \Delta S$. It is a natural function of C and S and this satisfies $\dot{\Pi} = \dot{C} - \dot{S}\Delta$ The conjugate variable Δ is a parameter which can be tuned, analagous to the chemical potential in a system with variable particle number. Show that if we choose $\Delta(t) = \frac{\partial C}{\partial S}$ and set $\dot{\Pi} = r\Pi$ we obtain the Black-Scholes equation

$$\frac{\partial C}{\partial t} + C^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC.$$

Is this equation deterministic or stochastic?