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Übungen zu Zeitabhängige Phänomene in der Statistischen Physik
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Übungsblatt 2: Hydrodynamics and viscous friction

Rotors on a lattice. In order to illustrate some of the essential features of Hydrodynamics we consider a simple model consisting of rigid rotors on a lattice. Each rotor can rotate freely without friction in the 2-dimensional x-y plane with a nearest neighbour interaction which tends to align the rotors. In this model there exists both a high temperature disordered phase and and low temperature ordered phase which are reminiscent of the paramagnetic and ferromagnetic phases of the Ising model. The model is sketched in the picture below.

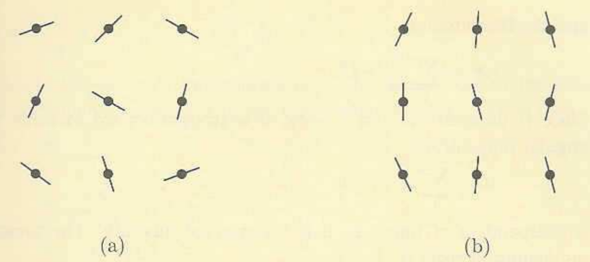

Fig. 8.2.1. (a) Disordered phase of the rigid rotor model. Rigid rotors at lattice sites l rotate freely about a fixed axis. The direction of the rotors is random. (b) Ordered phase of the rigid rotor model. Rotors align along a common axis as in a nematic liquid crystal. The order parameter $\langle Q_{ij} \rangle$ is nonzero.

1. The appropriate thermodynamic potential is given by $W(T, \Omega) = E - \Omega L - TS$, where T, Ω, E, L and S are the temperature, angular velocity, energy, total angular momentum and entropy, respectively. Discuss why this is the correct potential to use in this case. Explain why this implies the thermodynamic identity $Tds = d\epsilon - \Omega dl$, where s, ϵ and l are entropy, energy and angular momentum densities. Hydrodynamic variables obey conservation equations and therefore exhibit temporal freqencies $\sim q$ for small wavevectors. Identify the conserved densities (hydrodynamic variables) for this model and derive the following equation relating the change in entropy to changes in these variables

$$T\frac{ds}{dt} = -\nabla \cdot (\mathbf{j}^\epsilon - \Omega \mathbf{j}^l) \;\; - \mathbf{j}^l \cdot \nabla \Omega,$$

where $\mathbf{j}^\epsilon$ and $\mathbf{j}^l$ are the currents. By assuming the boundary condition that the entropy/heat current $\mathbf{Q} = \mathbf{j}^\epsilon - \Omega \mathbf{j}^l$ is zero on the boundaries of some large volume V show that

$$T\frac{dS}{dt} = \int_V d\mathbf{r} \;\left[-\mathbf{Q} \cdot \left(\frac{\nabla T}{T} \right) - \mathbf{j}^l \cdot \nabla \Omega \right].$$

This equation gives the rate of entropy production in the system.

2. Consider first the situation when there is no dissipation in the system and the entropy remains constant. What does this imply for the currents $\mathbf{Q}$ and $\mathbf{j}^l$? With dissipation the rate of entropy production must be positive by the second law of thermodynamics. In order to treat the dissipative case we assume the following *constitutive relations*

$$\mathbf{Q} = -\kappa \nabla T \ , \ \ \mathbf{j}^l = -\Gamma \nabla \Omega.$$

Why are these relations physically reasonable and under what conditions might they break down? Why must we require $\kappa > 0, \Gamma > 0$? There are no constitutive relations coupling $\mathbf{j}^l$ to ∇T or $\mathbf{Q}$ to $\nabla \Omega$ because these quantities have the same sign under time reversal. Finally, show that the diffusive equations for the energy and angular momentum densities linearised about $\Omega = 0$ are given by

$$\frac{\partial \epsilon}{\partial t} = D_\epsilon \nabla^2 \epsilon \qquad\qquad \frac{\partial l}{\partial t} = D_l \nabla^2 l,$$

where $D_\epsilon = \kappa / C_l$, $D_l = \Gamma / I$ and we have used the thermodynamic relations $\left(\frac{\partial l}{\partial \Omega}\right)_T = I$ and $\left(\frac{\partial \epsilon}{\partial T}\right)_l = C_l$. These equations are the phenomenological diffusion equations for the conserved densities. Analogous equations exist for the conjugate fields T and Ω which describe thermal and angular velocity diffusion.

Viscous friction. The shear viscosity η determines the friction one feels when stirring fluids at not too large velocities. A simple calculation considers an incompressible fluid described by the Navier-Stokes equations bounded by a moving wall. Let the wall be at position $z = 0$ and move parallel to its orientation, viz. $\mathbf{u}_{\text{wall}} = u(t)\hat{\mathbf{x}}$. Find the viscous force per area $F(t)$ required to move the wall with velocity $u(t)$ under the assumption of laminar (non-turbulent) flow, which states that the fluid velocity depends on the distance to the wall only, $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(z, t)$. *Hint:* Make the most simple possible assumption for the boundary condition of the fluid velocity at the wall. Then consider a single Fourier-mode for the wall velocity, $u(t) = u_\omega \cos \omega t$, and solve for the resulting friction at the wall, F_ω. Why does the term 'viscous skin effect' apply? It is an interesting slightly involved excercise to obtain $F(t)$ by inverse Fourier-transformation. The result is

$$F(t) = -\sqrt{\frac{\eta\rho}{\pi}} \int_{-\infty}^{t} ds \ \frac{\dot{u}(s)}{\sqrt{t-s}}$$

Discuss the long time behavior of $F(t)$ for the case of a wall initially at rest and then set to motion with constant velocity u_0 at time $t = 0$.