UNIVERSITÄT KONSTANZ
Fachbereich Physik
Prof. Dr. Matthias Fuchs
Raum P 907, Tel. (07531)88-4678
E-mail: matthias.fuchs@uni-konstanz.de

**Übungen zur Zeitabhängige Phänomene in der Statistischen Physik
Sommer Semester 2006**

Übungsblatt 1: Ferrohydrodynamics

Ferromagnetism is observed in many solid metals (e.g. iron or nickel) at temperatures below the Curie temperature T_{curie}. As the temperature of a ferromagnetic sample is increased it first undergoes a transition to a paramagnetic solid at T_{curie} and then melts to form a liquid at some higher temperature T_{melt}. As $T_{\text{melt}} > T_{\text{curie}}$ ferromagnetic fluids (ferrofluids) are not found in nature. However, such ferrofluids can be synthesized by suspending magnetic colloidal particles in a solvent. Each colloidal particle has a solid, single domain magnetic core and acts like a small magnet. If an external magnetic field H is applied then the particles within a small volume element of the fluid can align to give an induction field $B(\mathbf{r})$, where $\mathbf{r}$ is the center of the element. The presence of a B field density within the fluid adds additional terms to the standard hydrodynamic equations which lead to interesting and unusual flow behaviour.
In classical hydrodynamics the density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$ are fields which can vary in space and time. These functions may then, in principle, be found from the Navier-Stokes equations. We will consider the derivation of the analogous equations for an incompressible ferrofluid.

1. **Entropy equation**. For a system with total momentum $\mathbf{G}$ and velocity $\mathbf{v}$ the grand potential is given by $-\Omega(\mu, V, T, \mathbf{v}) = -E + \mu N + TS + \mathbf{G} \cdot \mathbf{v}$. Taking differentials of this expression leads to the entropy equation (see problem 58 in Statistical Mechanics 1). Consider how the entropy equation is modified for a ferrofluid in an external field.

2. **Magnetic stress-tensor** In a normal non-magnetic fluid viscous forces enter the equations of motion via the stress tensor. However, in a ferrofluid we must also consider the contribution of an additional magnetic stress tensor. The diagonal elements reflect the influence of a magnetic energy density within a fluid element on the pressure. Consider the meaning of the off diagonal elements. When there are also electric charges present it is appropriate to use the Maxwell stress tensor, given by

$$T_{\alpha\beta}^{\text{maxwell}} = \epsilon_0 (E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B})\delta_{\alpha\beta}).$$

   Derive this expression using Maxwells equations, the Lorentz force and Newtons principle (action = reaction). Expressing the force as a spatial integral over a force density allows connection to be made to the divergence of the Maxwell stress tensor. When $\mathbf{E} = 0$ we obtain the magnetic stress tensor $\mathbf{T}_m$.

3. **Equation of motion** Consider a cubic volume element in the ferrofluid with volume $dV = dxdydz$, large enough to contain many particles but small compared to variations in the density and velocity fields. Express the momentum of this element as a function of dV, ρ and $\mathbf{v}$. By considering a volume element which *moves with the flow* argue that Newtons law becomes

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{f}_g + \mathbf{f}_p + \mathbf{f}_v + \mathbf{f}_m,$$

where the $\mathbf{f}_i$ are the gravitational, pressure, viscous and magnetic force densities. Think about the meaning of each term on the right hand side of this equation. The notation D/Dt indicates the convective (also known as the substantial or material) derivative. What is the convective derivative and why is it useful here?

4. **Navier-Stokes equation** Give expressions for $\mathbf{f}_g$ and $\mathbf{f}_p$ in terms of the acceleration due to gravity $\mathbf{g}$ and the pressure acting on the element $p(\rho, T)$. The viscous stress tensor $\mathbf{T}_v$ for an incompressible fluid ($\nabla \cdot \mathbf{v} = 0$) is related to the velocity by the Newtonion constitutive relation $\mathbf{T}_v = \eta[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$. By taking divergences of $\mathbf{T}_v$ and $\mathbf{T}_m$ find an expressions for the viscous and magnetic forces $\mathbf{f}_v$. Thus show that the Navier-Stokes equation for ferrofluids is given by

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p^* \, + \, \mu_0 M \nabla H \, + \, \eta \nabla^2 \mathbf{v} \, + \, \rho \mathbf{g},$$

where p^* is an effective pressure, modified by the presence of magnetic field energy within the element. Note that the term $\mu_0 M \nabla H$ comes from using Maxwell's relation $\nabla \cdot B = 0$, Ampère's law in the absence of current $\nabla \times \mathbf{H} = 0$ and assuming that $\mathbf{B}$ and $\mathbf{H}$ are parallel.

5. **Bernoulli equation** We now derive the Ferrohydrodynamic Bernoulli equation from the Navier-Stokes equation for an incompressible ferrofluid. First consider the viscous term $\eta \nabla^2 \mathbf{v}$ and show that for irrotational flow ($\nabla \times \mathbf{v} = 0$) the viscous term is zero. It can be assumed that the magnetization M is approximately constant in space such that $\nabla MH \approx M \nabla H$. Use the vector identity $\mathbf{v} \cdot \nabla \mathbf{v} = \nabla(\frac{1}{2}v^2) - \mathbf{v} \times (\nabla \times \mathbf{v})$ to obtain the ferrohydrodynamic Bernoulli equation

$$-\rho \frac{\partial \phi}{\partial t} + p^* + \frac{1}{2}\rho v^2 + \rho g h - \mu_0 M H = f(t),$$

where ϕ is the velocity potential ($\mathbf{v} = -\nabla \phi$) and $f(t)$ is a constant of integration. For steady state flows $\partial \phi / \partial t = 0$ and $f(t) = const.$