SSH Model

Alessandro David

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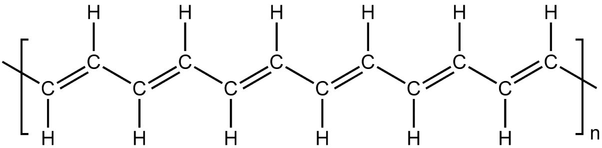
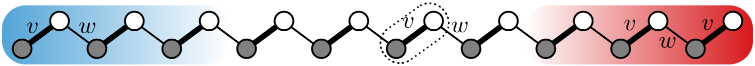
Adapted from Lecture Notes at:
https://arxiv.org/abs/1509.02295
and from article:
Nature Physics **9**, 795 (2013)

Motivations

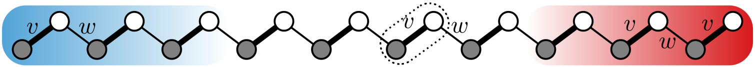
SSH = Su-Schrieffer-Heeger





- Polyacetylene molecule, staggered hopping
- Simplest 1D model presenting topological behaviour
- Introduction of many concepts of topological band theory

Defining Properties

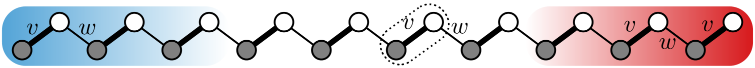


- *Finite* 1D lattice:

  $$N \text{ unit cells} \rightarrow 2 \text{ sites} \,/\, \text{unit cell} \rightarrow \begin{cases} 2N \text{ sites} \\ \text{sublattices } A \text{ and } B \end{cases}$$

  bulk + edges
- no spin
  - spin-polarized particles
  - for real systems: take two copies of it
- fermions (half-filling)
- hopping with staggered amplitudes: $v, w \geq 0$ (in case, redefine base states to cancel complex phases)
- no on-site potential

Hamiltonian



No interactions between particles ⇒ single particle hamiltonian:

$$H = v \sum_{m=1}^{N} (|m, A\rangle\langle m, B| + \text{h.c.}) + w \sum_{m=1}^{N-1} (|m+1, A\rangle\langle m, B| + \text{h.c.})$$

Separate internal and external d.o.f.: $|m, \alpha\rangle = |m\rangle \otimes |\alpha\rangle$,
$m = 1, \ldots, N$, $\alpha = A, B$:

$$H = v \sum_{m=1}^{N} (|m\rangle\langle m| \otimes \sigma_x + \text{h.c.}) + \\ + w \sum_{m=1}^{N-1} \left(|m+1\rangle\langle m| \otimes \frac{\sigma_x + i\sigma_y}{2} + \text{h.c.} \right)$$

Bulk Hamiltonian

Connect the edges with periodic (Born-von Karman) boundary conditions:

$$H_{\text{bulk}} = \sum_{m=1}^{N} \left(v|m\rangle\langle m| \otimes \sigma_x + w|(m \mod N) + 1\rangle\langle m| \otimes \sigma_+ + \text{h.c.} \right)$$

looking for the eigenstates: $H_{\text{bulk}}|\psi_n\rangle = E_n|\psi_n\rangle$, $n = 1, \ldots, 2N$.
We have translational symmetry in the cell index, m. Introduce plane wave solutions:

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{imk}|m\rangle, \quad k \in \{\delta, 2\delta, \ldots, N\delta\}, \quad \delta = \frac{2\pi}{N}$$

Total eigenstates: $|\psi_n(k)\rangle = |k\rangle \otimes |u_n(k)\rangle$, with:

$$|u_n(k)\rangle = a_n(k)|A\rangle + b_n(k)|B\rangle, \quad n = 1, 2$$

Bulk Momentum-Space Hamiltonian

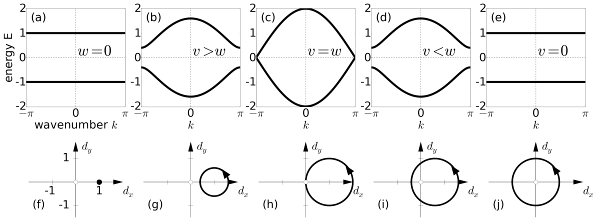
The vectors $|u_n(k)\rangle$ are the eigenstates of the bulk momentum-space hamiltonian:

$$H(k) = \langle k|H_{\text{bulk}}|k\rangle, \qquad H(k)|u_n(k)\rangle = E_n(k)|u_n(k)\rangle$$

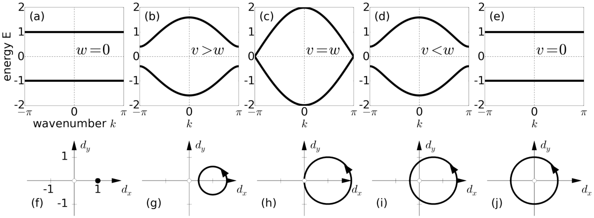
$$H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix}$$

Then $H(k)^2 = |v + we^{ik}|^2 \mathbb{1} = E(k)^2 \mathbb{1}$, so:

$$E_\pm(k) = \pm|v + we^{ik}| = \pm\sqrt{v^2 + w^2 + 2vw\cos k}$$

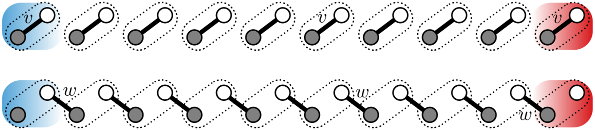


Bulk Winding Number



- $H(k) = d_0(k)\sigma_0 + d_x(k)\sigma_x + d_y(k)\sigma_y + d_z(k)\sigma_z$
- $d_0(k) = 0, d_x(k) = v + w\cos k, d_y(k) = w\sin k, d_z(k) = 0$
- magnitude: $|\mathbf{d}(k)| = E(k)$
- direction of $\mathbf{d}(k)$ describes the internal structure of $|u_n(k)\rangle$
- as k goes from 0 to 2π, $\mathbf{d}(k)$ traces a circle of radius w centered at $(v, 0)$
- the bulk winding number ν is defined by the number of times this path goes around the origin $(0, 0)$
- for SSH, $\nu = 0$ if $v > w$, $\nu = 1$ if $v < w$

Edge States (I): Fully Dimerized



In the fully dimerized limits the eigenstates are restricted to one dimer each.

- case $v = 1, w = 0$ is in the *trivial* phase ($\nu = 0$):

$$H\left(|m, A\rangle \pm |m, B\rangle\right) = \pm\left(|m, A\rangle \pm |m, B\rangle\right)$$

  moreover $H(k) = \sigma_x$ (k independent)
- case $v = 0, w = 1$ is in the *topological* phase ($\nu = 1$):

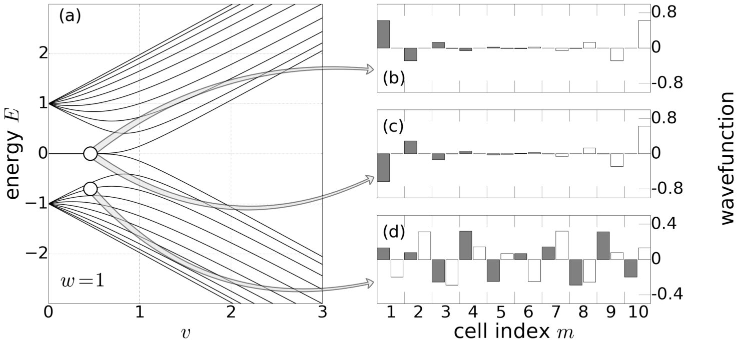
$$H\left(|m, B\rangle \pm |m+1, A\rangle\right) = \pm\left(|m, B\rangle \pm |m+1, A\rangle\right)$$

  moreover $H(k) = \sigma_x \cos k + \sigma_y \sin k$. In this case we also have *zero energy eigenstates*:

$$H|1, A\rangle = H|N, B\rangle = 0$$

  they have support only on one site and only at the edge

Edge State (II): Simulation



Chiral Symmetry (I)

- Crash course on symmetry and quantum mechanics:
  - H (hermitian) has symmetry U (unitary) iff:

$$UHU^\dagger = H \quad \leftrightarrow \quad [H, U] = 0$$

  - in case of symmetry, H and U can be diagonalized simultaneously
- We talk about symmetry even when we have an operator Γ such that:

$$\Gamma H \Gamma^\dagger = -H \quad \leftrightarrow \quad \{H, \Gamma\} = 0$$

  this is called *chiral symmetry*.
- We require Γ to be:
  - *unitary* and *hermitian*: $\Gamma^\dagger \Gamma = \Gamma^2 = \mathbb{1}$
  - *local*: it acts only inside unit cell
  - *robust*: $\Gamma H(\xi) \Gamma = -H$, $\forall \xi \in \Xi$

Chiral Symmetry (II): General Consequences

- *Orthogonal sublattice projectors*:

$$P_A = \frac{1}{2}(\mathbb{1} + \Gamma) \quad P_B = \frac{1}{2}(\mathbb{1} - \Gamma) \quad P_A + P_B = \mathbb{1} \quad P_A P_B = 0$$

- $\Gamma H \Gamma = -H$ is equivalent to $P_A H P_A = P_B H P_B = 0$
- the spectrum is *symmetric*:

$$H|\psi\rangle = E|\psi\rangle \implies H\Gamma|\psi\rangle = -\Gamma H|\psi\rangle = -E|\psi\rangle$$

- if $E \neq 0$, $|\psi\rangle$ and $\Gamma|\psi\rangle$ are orthogonal, then:

$$0 = \langle\psi|\Gamma|\psi\rangle = \langle\psi|P_A|\psi\rangle - \langle\psi|P_B|\psi\rangle$$

- if $E = 0$, $|\psi\rangle$ and $\Gamma|\psi\rangle$ are in the same eigenspace, so zero energy eigenstates may have support only on one sublattice:

$$HP_{A/B}|\psi\rangle = H(|\psi\rangle \pm \Gamma|\psi\rangle)/2 = 0$$

Chiral Symmetry (III): Consequences for SSH

- SSH hamiltonian is *bipartite*: it has no transitions in the same sublattice
- the projectors are then:

$$P_A = \sum_{m=1}^{N} |m, A\rangle\langle m, A| \qquad P_B = \sum_{m=1}^{N} |m, B\rangle\langle m, B|$$

- the chiral symmetry operator is then:

$$\Gamma = \Sigma_z = P_A - P_B = \bigoplus_{m=1}^{N} \sigma_z$$

- H contains terms proportional to $|m, A\rangle\langle m', B|$ or $|m, B\rangle\langle m', A|$, so Σ_z is robust

Computing Winding Numbers

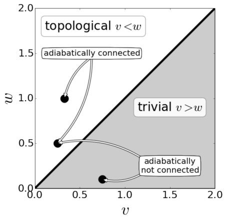
- *Graphically*:
  - $\mathbf{d}(k)$ is a directed line with two sides (left and right)
  - take a directed line $\mathcal{L}$ from origin to infinite
  - assign $+1$ every time $\mathbf{d}(k)$ encounters $\mathcal{L}$ with the left side and -1 when this happens with the right side
  - winding number ν is the sum of the assigned numbers.

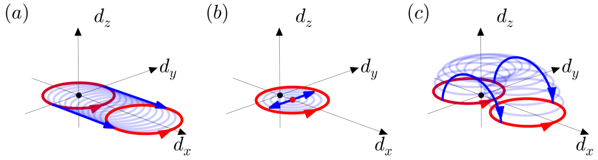
  when $\mathcal{L}$ or $\mathbf{d}(k)$ are deformed continuously, intersections between the two can only appear or disappear pairwise with signatures $+1$ and -1, leaving ν unchanged and *topologically invariant*
- by *integral*:

$$\widetilde{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|} \qquad \nu = \frac{1}{2\pi} \int \left(\widetilde{\mathbf{d}}(k) \times \frac{d}{dk} \widetilde{\mathbf{d}}(k) \right)_z dk$$

Topological Invariants

- An insulating hamiltonian is *adiabatically deformed* if:
  - its parameters are changed continuously
  - the symmetries are preserved
  - the bulk gap around $E = 0$ remains open
- two insulating hamiltonians are said to be *adiabatically connected* or *adiabatically equivalent* if there is an adiabatic deformation connecting them.
- an integer number characterizing an insulating hamiltonian that does not change under adiabatic transformations is a *topological invariant*





Number of Edge States as Topological Invariant

- Consider eigenstates of a chiral symmetric 1D hamiltonian at the left edge in the thermodynamic limit, with energies laying in $-\varepsilon < E < \varepsilon$ in the bulk
- the number of zero energy state is finite: N_A on sublattice A and N_B on sublattice B
- consider an adiabatic deformation controlled by parameter $p : 0 \to 1$ and its effect on $N_A - N_B$ $(E_{\text{bulk gap}} > 2\varepsilon, \forall p)$
  - if a non-zero energy edge state is brought to zero energy, its chiral partner does the same, so $N_A - N_B$ is unchanged
  - opposite: if a zero energy edge state is brought to $E > 0$, it must have a $E < 0$ chiral partner
  - if non-zero energy eigenstates enter or exit the $-\varepsilon < E < \varepsilon$ window, this has no effect on $N_A - N_B$
  - edge states with energies in the gap must decay exponentially, so the deformation can extend the support of edge state but only by a limited amount

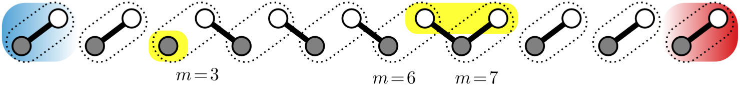
Bulk-Boundary Correspondence for SSH

- The winding number ν, a property of the bulk, is a topological invariant and it is 0 when $v > w$, 1 when $v < w$.
- The net number of edge states at the left (or right) edge $N_A - N_B$, a property of the boundary, is also a topological invariant and it has the same values of ν for every choice of v, w.

Bound States at Domain Walls

Edge states occurs at:

- edges of a chain
- walls between domains with different parameters



In the fully dimerized case we have two possibilities:

- single isolated sites which host zero energy states
- trimers where odd superpositions of the two end sites have zero energy:

$$H(|m,\alpha\rangle - |m+n,\alpha\rangle) = 0$$

Calculation of SSH Edge States (I)

$$H = \sum_{m=1}^{N} (v_m |m, A\rangle\langle m, B| + \text{h.c.}) + \sum_{m=1}^{N-1} (w_m |m+1, A\rangle\langle m, B| + \text{h.c.})$$

We are looking for a_m and b_m, $m = 1, \ldots, N$, for zero energy eigenstates, such that:

$$H \sum_{m=1}^{N} (a_m |m, A\rangle + b_m |m, B\rangle) = 0$$

for these $2N$ equations we have the solutions:

$$a_m = \prod_{j=1}^{m-1} \frac{-v_j}{w_j} a_1 \qquad m = 2, \ldots, N$$

$$b_m = \frac{-v_N}{w_m} \prod_{j=m+1}^{N-1} \frac{-v_j}{w_j} b_N \qquad m = 1, \ldots, N-1$$

but also $b_1 = a_N = 0$, so no zero energy eigenstates.

Calculation of SSH Edge States (II)

We can find two approximate solutions for $N \to \infty$. Define v, w such that:

$$\overline{\log|v|} = \frac{1}{N-1}\sum_{m=1}^{N-1} \log|v_m|; \qquad \overline{\log|w|} = \frac{1}{N-1}\sum_{m=1}^{N-1} \log|w_m|;$$

so $b_1 = a_N = 0$ become:

$$|a_N| = |a_1| e^{-(N-1)/\xi}; \qquad |b_1| = |b_N| e^{-(N-1)/\xi} \frac{v_N}{w_1}$$

with $\xi = 1/(\overline{\log|w|} - \overline{\log|v|})$. Approximate zero energy solutions:

$$|L\rangle = \sum_{m=1}^{N} a_m |m, A\rangle; \qquad |R\rangle = \sum_{m=1}^{N} b_m |m, B\rangle;$$

In case, v_m and w_m are all real and equal: $\xi = 1/\log(w/v)$

Calculation of SSH Edge States (III)

Under H, states $|L\rangle$ and $|R\rangle$ overlap by an exponentially decaying amount (being only approximate eigenstates) and then hybridize:

$$\langle R|H|L\rangle = \left|a_1 v_N b_N e^{-(N-1)/\xi}\right| e^{i\phi}, \quad \phi \in [0, 2\pi)$$

$$|0_\pm\rangle = \frac{|L\rangle \pm e^{i\phi}|R\rangle}{\sqrt{2}}; \qquad E_\pm = \pm \left|a_1 v_N b_N e^{-(N-1)/\xi}\right|$$

Experimental Realization (I)

nature physics

ARTICLES

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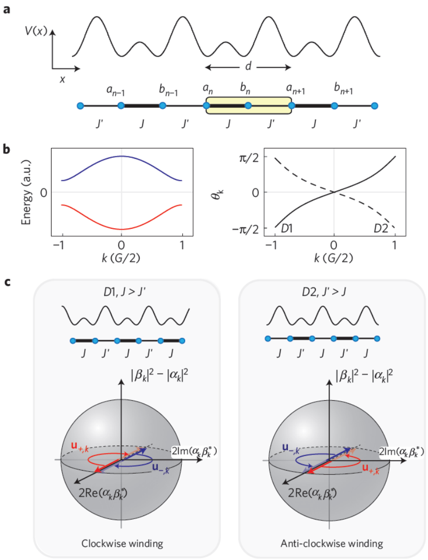
Direct measurement of the Zak phase in topological Bloch bands

Marcos Atala[1†], Monika Aidelsburger[1†], Julio T. Barreiro[1,2], Dmitry Abanin[3], Takuya Kitagawa[3,4], Eugene Demler[3] and Immanuel Bloch[1,2★]

Geometric phases that characterize the topological properties of Bloch bands play a fundamental role in the band theory of solids. Here we report on the measurement of the geometric phase acquired by cold atoms moving in one-dimensional optical lattices. Using a combination of Bloch oscillations and Ramsey interferometry, we extract the Zak phase—the Berry phase gained during the adiabatic motion of a particle across the Brillouin zone—which can be viewed as an invariant characterizing the topological properties of the band. For a dimerized lattice, which models polyacetylene, we measure a difference of the Zak phase $\delta\varphi_{\mathrm{Zak}} = 0.97(2)\pi$ for the two possible polyacetylene phases with different dimerization. The two dimerized phases therefore belong to different topological classes, such that for a filled band, domain walls have fractional quantum numbers. Our work establishes a new general approach for probing the topological structure of Bloch bands in optical lattices.

Zak phase: $\varphi_{\mathrm{Zak}} = i \int \langle u_k | \partial_k | u_k \rangle dk$

Experimental Realization (II)



Experimental Realization (III)