$\mathbf{Z}_2$ invariant for time reversal two dimensional topological insulators

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Plan

1)Topologically trivial and nontrivial phases of time reversal topological insulator-definition of Z_2-invariant

2)How to calculate Z_2-invariant

3) BHZ model as an example of different types of topological behaviour

4) Z_2-invariant in real physical systems

5) Alternative ways to introduce topological invariant

Literature

1) J.K.Asboth, L. Orozlany, A. Palyi
Lecture_Notes_arXiv:1509.02295

2) Liang Fu, C.L. Kane
Time reversal polarisation and a Z_2 adiabatic spin pump

Phys. Rev. B 74, 195312

3)Rui Yu, Xiao Liang Qi, Andrei Bernevig, Zhong Fang, Xi Dai
Equivalent expression of Z_2 topological invariant for band insulators using the non-Abelian Berry connection

Phys. Rev. B 84, 075119

Topologically trivial and nontrivial phases of time reversal topological insulator-definition of Z_2-invariant

1) We start with a bulk Hamiltonian $\hat{H}(k_x, k_y)$. Reinterpret k_y as time: $\hat{H}(k_x, k_y \to t)$ is a bulk 1D Hamiltonian of an adiabatic pump.
2) We sweep the time and track the motion of the particles, that we associate with the Wannier centres flow.
3) If an adiabatic change to the Hamiltonian exists that would turn off the pump (the Wannier centres don't move), we say that the topological insulator is in the trivial phase(Z_2=0), if no such change exists-then it's in the nontrivial phase(Z_2=1)

How to calculate Z_2-invariant

$$|k_x >= \frac{1}{\sqrt{N}} \sum_m e^{imk_x} |m > \ , \quad m \in 1, 2, ..., N$$

$$|\psi_n(k_x) >= |k_y > \otimes |k_x > \otimes |u_n(k_x, k_y)$$

$$n \in 1, 2, ..., N_F$$

$$\hat{X} = \sum_m e^{2\pi i \frac{m}{N}} |m >< m|$$

$$\hat{P} = \sum_{n,k_x} |\psi_n(k_x) >< \psi_n(k_x)|$$

$$\hat{X}_p = \hat{P} \hat{X} \hat{P}$$

Wannier functions W(j,n) are eigenfunctions of $\hat{X}_p$ with eigenvalues $\lambda(n, j)$. The position of the centre of the Wannier function can be associated with $\frac{N}{2\pi} Im \log(\lambda(n, j))$.

How to calculate Z_2-invariant

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$$\hat{X}_p = \hat{P}\hat{X}\hat{P}$$

$$\hat{X}_p = \hat{P}\hat{X}\hat{P} = \sum_{n,k_x} \sum_{\tilde{n},\tilde{k}_x} < \psi_n(k_x)|\hat{X}|\psi_{\tilde{n}}(\tilde{k}_x) > |\psi_n(k_x) >< \psi_{\tilde{n}}(\tilde{k}_x)|$$

How to calculate Z_2-invariant

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$$|\psi_n(k_x) >= |k_x > \otimes |u_n(k_x) >$$

$$|k_x >= \frac{1}{\sqrt{N}} \sum_m e^{imk_x} |m >$$

$$\hat{X} = \sum_m e^{2\pi i \frac{m}{N}} |m >< m| = \sum_m e^{i\delta k m} |m >< m|$$

How to calculate Z_2-invariant

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$$|k_x >= \frac{1}{\sqrt{N}} \sum_m e^{imk_x} |m >$$

$$\hat{X} = \sum_m e^{2\pi i \frac{m}{N}} |m >< m| = \sum_m e^{i\delta k m} |m >< m|$$

$$\hat{X}_p = \sum_{n,\tilde{n},k_x} < u_n(k_x + \delta k)|u_{\tilde{n}}(k_x) > |\psi_n(k_x + \delta k_x) >< \psi_{\tilde{n}}(k_x)|$$

How to calculate Z_2-invariant

$$|\psi_n(k_x)>=|k_x>\otimes|u_n(k_x)>$$

$$\hat{X}_p=\sum_{n,\tilde{n},k_x}<u_n(k_x+\delta k)|u_{\tilde{n}}(k_x)>|\psi_n(k_x+\delta k_x)><\psi_{\tilde{n}}(k_x)|$$

$$(\hat{X}_p)^N=\sum_{m,\tilde{n},k_x}W_{m\tilde{n}}(k_x)|\psi_m(k_x)><\psi_{\tilde{n}}(k_x)|$$

$$W_{m,\tilde{n}}(k_x)=<u_m(k_x)|u_r(k_x+(N-1)\delta k)>*$$
$$*<u_r(k_x+(N-1)\delta k)|u_h(k_x+(N-2)\delta k)>......<u_n(k_x+\delta k)|u_{\tilde{n}}(k_x)>$$

Eigenvalues of the Wilson loop $W(k_X)$ don't depend on k_X

$$W(k_x)=ABCD$$

$$W(k_x+\delta k)=DABC$$

$$ABCD*\mathbf{v}=\lambda*\mathbf{v}$$

$$DABC*D\mathbf{v}=D*ABCD*\mathbf{v}=\lambda D\mathbf{v}$$

How to calculate Z_2-invariant

$$(\hat{X}_p)^N = \sum_{m,\tilde{n},k_x} W_{m\tilde{n}}(k_x)|\psi_m(k_x) >< \psi_{\tilde{n}}(k_x)|$$

$$W_{m,\tilde{n}}(k_x) =< u_m(k_x)|u_r(k_x + (N-1)\delta k) > *$$
$$* < u_r(k_x + (N-1)\delta k)|u_h(k_x + (N-2)\delta k) > ...... < u_n(k_x + \delta k)|u_{\tilde{n}}(k_x) >$$

Eigenvalues of the Wilson loop W(k_X) don't depend on k_X

$$W(k_x)\mathbf{v}(k_x) = \lambda_i \mathbf{v}(k_x),\; i \in 1, ...N_F$$

$$(\hat{X}_p)^N(\sum_{k_x} \mathbf{v}(k_x)) = \lambda_i(\sum_{k_x} \mathbf{v}(k_x))$$

Then $\hat{X}_p$ has $N * N_f$ eigenvalues:

$$\tilde{\lambda}_i(j) = |\lambda_i|^{1/N} e^{(\frac{2\pi}{N}\Theta_i + 2\pi\frac{j}{N})}$$

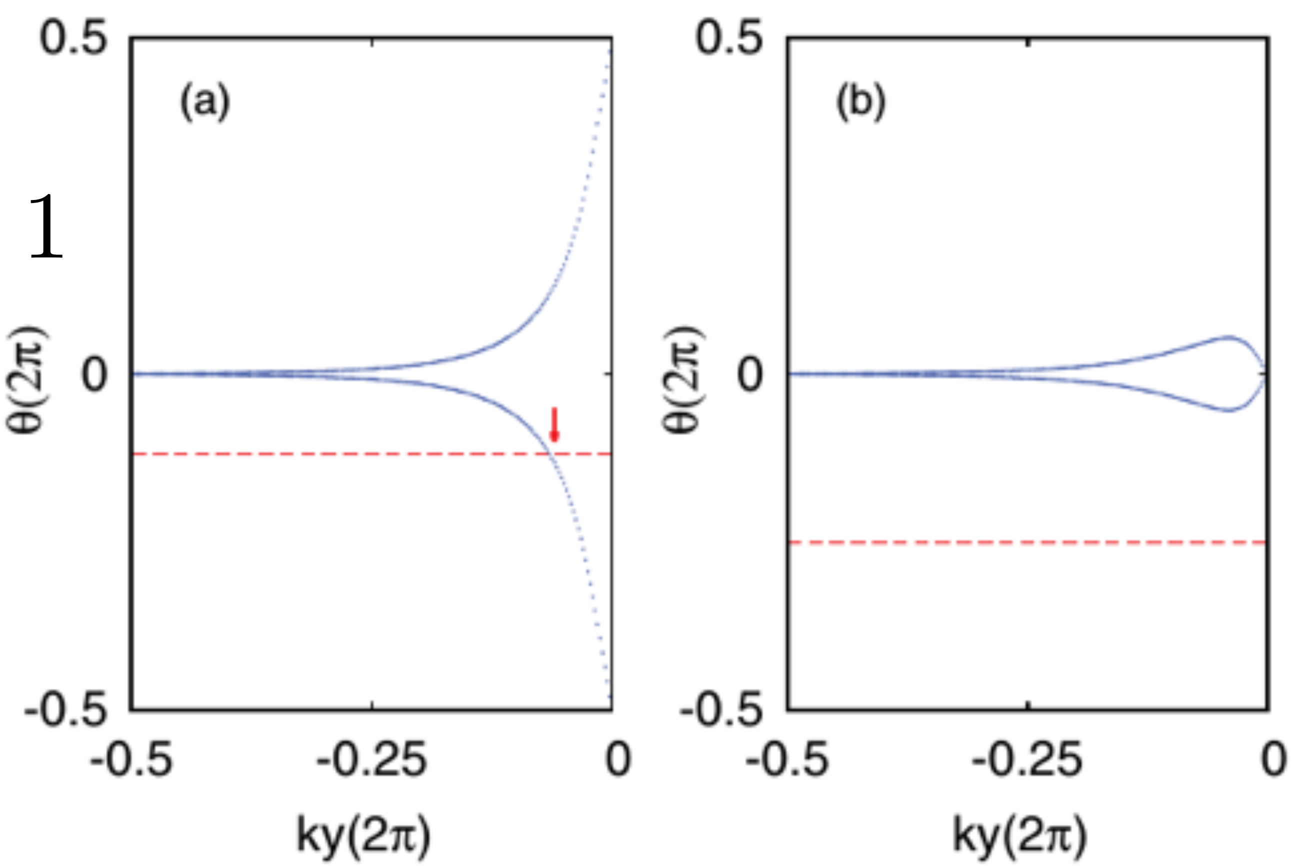
$\frac{N}{2\pi} Im \log \tilde{\lambda}_i(j) = \Theta_i + j$ Positions of the Wannier centres

Example, BHZ model

$$H_{\text{eff}}(k_x, k_y) = \begin{bmatrix} H(\mathbf{k}) & 0 \\ 0 & H^*(-\mathbf{k}) \end{bmatrix},$$

where $H(\mathbf{k}) = \varepsilon(\mathbf{k}) + d_i(\mathbf{k})\sigma_i$, $d_1 + id_2 = Aa^{-1}[\sin k_x a + i \sin k_y a]$, $d_3 = -2Ba^{-2}[2 - \frac{M}{2B} - \cos k_x a - \cos k_y a]$, and $\varepsilon(\mathbf{k}) = C - 2Da^{-2}[2 - \cos k_x a - \cos k_y a]$.

Z_2 invariant is 1

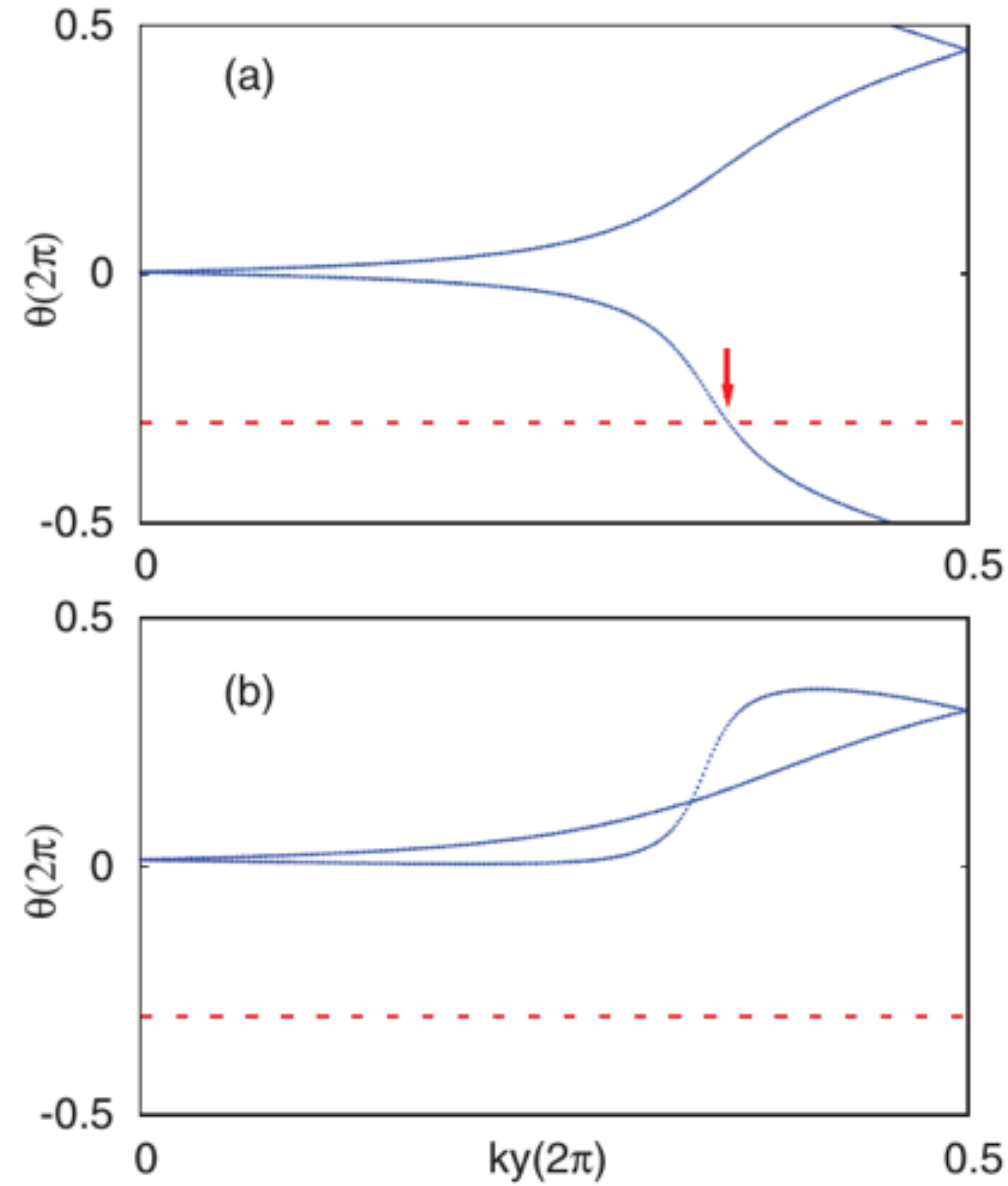


Z_2 invariant is 0

Z_2 invariant in Graphen

$$H = t \sum_{\langle i,j\rangle,\sigma} c_{i,\sigma}^{\dagger} c_{j,\sigma} + i\lambda_{\mathrm{so}} \sum_{\langle\langle i,j\rangle\rangle,\sigma\sigma'} \nu_{ij} c_{i,\sigma}^{\dagger} s_{\sigma\sigma'}^{z} c_{j,\sigma'}$$
$$+ i\lambda_R \sum_{\langle i,j\rangle,\sigma\sigma'} c_{i,\sigma}^{\dagger} (\mathbf{s}_{\sigma\sigma'} \times \hat{\mathbf{d}}_{ij})_z c_{j,\sigma'} + \lambda_v \sum_{i,\sigma} \xi_i c_{i,\sigma}^{\dagger} c_{i,\sigma},$$

Changing the parameter λ_v of the Hamiltonian it's possible to switch between different topological regimes



Alternative ways to define Z_2 invariant

1) We could as well calculate the number of edge state(Kramers pairs of edge states)

2) It turns out that adiabatic deformation of the Hamiltonian can only destroy pairs of Kramers states, which means that the parity of the number of edge states is also a topological number

3)Theorem: Invariant defined in such a way is equivalent to the one we defined

Conclusion

1) We introduced the Z_2 invariant for time reversal topological insulators, which defines whether the material will exhibit nontrivial topological behaviour or not.
2) A general procedure to calculate the invariant was shown
3) The method was tested on real system of graphene