Time reversal invariant 2d lattice models

Student seminar on topoligical insulators

University of Konstanz

Florian Ginzel

December 15, 2016

General framework

- Chern insulators host chiral edge states $\Rightarrow$ not time reversal invariant (TRI)

General framework

- Chern insulators host chiral edge states $\Rightarrow$ not time reversal invariant (TRI)
- Can construct TRI system from given Hamiltonian H:

$$H_{\mathrm{TRI}} = \begin{pmatrix} H & C \\ C^\dagger & H^* \end{pmatrix}$$

General framework

- Chern insulators host chiral edge states $\Rightarrow$ not time reversal invariant (TRI)
- Can construct TRI system from given Hamiltonian H:

$$H_{\mathrm{TRI}} = \begin{pmatrix} H & C \\ C^\dagger & H^* \end{pmatrix}$$

- Define $\quad s_i = \sigma_i \otimes \mathbb{1}_{\mathrm{ext}} \otimes \mathbb{1}_{\mathrm{int}}$, $\sigma_i \in \{0, x, y, z\}$

$$\begin{array}{lll} \mathcal{T}_1 = i s_y K & \text{with} & \mathcal{T}_1^2 = -\mathbb{1} \\ \mathcal{T}_2 = s_x K & \text{with} & \mathcal{T}_2^2 = \mathbb{1} \end{array}$$

General framework

- Chern insulators host chiral edge states $\Rightarrow$ not time reversal invariant (TRI)
- Can construct TRI system from given Hamiltonian H:

$$H_{\mathrm{TRI}} = \begin{pmatrix} H & C \\ C^\dagger & H^* \end{pmatrix}$$

- Define $\quad s_i = \sigma_i \otimes \mathbb{1}_{\mathrm{ext}} \otimes \mathbb{1}_{\mathrm{int}}$, $\sigma_i \in \{0, x, y, z\}$

$$\begin{array}{lll} \mathcal{T}_1 = i s_y K & \text{with} & \mathcal{T}_1^2 = -\mathbb{1} \\ \mathcal{T}_2 = s_x K & \text{with} & \mathcal{T}_2^2 = \mathbb{1} \end{array}$$

- Coupling C determines type of time reversal symmetry:

$$\begin{array}{ll} C^T = -C & \Rightarrow [H, \mathcal{T}_1] = 0 \\ C^T = C & \Rightarrow [H, \mathcal{T}_2] = 0 \end{array}$$

General framework

- Chern insulators host chiral edge states $\Rightarrow$ not time reversal invariant (TRI)
- Can construct TRI system from given Hamiltonian H:

$$H_{\mathrm{TRI}} = \begin{pmatrix} H & C \\ C^\dagger & H^* \end{pmatrix}$$

- Define $\quad s_i = \sigma_i \otimes \mathbb{1}_{\mathrm{ext}} \otimes \mathbb{1}_{\mathrm{int}}, \ \sigma_i \in \{0, x, y, z\}$

$$\begin{array}{lll} \mathcal{T}_1 = i s_y K & \text{with} & \mathcal{T}_1^2 = -\mathbb{1} \\ \mathcal{T}_2 = s_x K & \text{with} & \mathcal{T}_2^2 = \mathbb{1} \end{array}$$

- Coupling C determines type of time reversal symmetry:

$$\begin{array}{ll} C^T = -C & \Rightarrow [H, \mathcal{T}_1] = 0 \\ C^T = C & \Rightarrow [H, \mathcal{T}_2] = 0 \end{array}$$

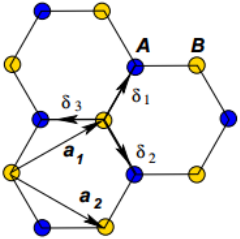
- C acts on internal degree of freedom (DOF) (e.g. sublattice, spin,...)

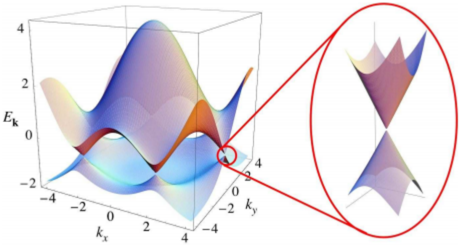
Graphene

- 2d honeycomb lattice
- Sublattice degree of freedom
- Dirac cones in the band structure
- Valley degree of freedom:

$$\mathbf{K} = \frac{2\pi}{3a}\left(1, \frac{1}{\sqrt{3}}\right)$$
$$\mathbf{K}' = \frac{2\pi}{3a}\left(1, -\frac{1}{\sqrt{3}}\right)$$

- No band gap





Graphics taken from A. H. Castro Neto, F. Guinea, N. M. R Peres, K. S. Novoselov, A. K. Geim: The electronic properties of graphene. In: Arxiv preprint. 2007, arxiv:0709.1163v2

The Kane and Mele model (I)

Idea: doubling the Haldande-model by introducing spin

Start from low energy Hamitonian near the Dirac cones at $\mathbf{K}$ and $\mathbf{K}'$ ($\phi = 0$)

$$H_0 = -i\hbar v_F \Psi^\dagger \left(\sigma_x \tau_z \partial_x + \sigma_y \partial_y\right) \Psi$$

with Pauli matrices σ, τ corresponding to sublattice and valley

The Kane and Mele model (I)

Idea: doubling the Haldande-model by introducing spin

Start from low energy Hamitonian near the Dirac cones at $\mathbf{K}$ and $\mathbf{K}'$ ($\phi = 0$)

$$H_0 = -i\hbar v_F \Psi^\dagger \left(\sigma_x \tau_z \partial_x + \sigma_y \partial_y\right) \Psi$$

with Pauli matrices σ, τ corresponding to sublattice and valley

Add spin $\mathbf{s}$, obtain (for $\tau_z = 1$, i.e. around $\mathbf{K}$)

$$H_0(\mathbf{K} + \mathbf{k}) = \Psi^\dagger \begin{pmatrix} h_\uparrow(\mathbf{k}) & 0 \\ 0 & h_\downarrow(\mathbf{k}) \end{pmatrix} \Psi$$

The Kane and Mele model (I)

Idea: doubling the Haldande-model by introducing spin

Start from low energy Hamitonian near the Dirac cones at $\mathbf{K}$ and $\mathbf{K}'$ ($\phi = 0$)

$$H_0 = -i\hbar v_F \Psi^\dagger \left(\sigma_x \tau_z \partial_x + \sigma_y \partial_y\right) \Psi$$

with Pauli matrices σ, τ corresponding to sublattice and valley

Add spin $\mathbf{s}$, obtain (for $\tau_z = 1$, i.e. around $\mathbf{K}$)

$$H_0(\mathbf{K} + \mathbf{k}) = \Psi^\dagger \begin{pmatrix} h_\uparrow(\mathbf{k}) & 0 \\ 0 & h_\downarrow(\mathbf{k}) \end{pmatrix} \Psi$$

Introduce TRI spin-orbit-coupling $H_{\mathrm{SO}} = \lambda_{\mathrm{SO}} \Psi^\dagger \sigma_z \tau_z s_z \Psi$

Doubly degenerate bands with a gap: $E_k = \pm\sqrt{(\hbar v_F k)^2 + \lambda_{\mathrm{SO}}}$

The Kane and Mele model (I)

Idea: doubling the Haldande-model by introducing spin

Start from low energy Hamitonian near the Dirac cones at $\mathbf{K}$ and $\mathbf{K}'$ ($\phi = 0$)

$$H_0 = -i\hbar v_F \Psi^\dagger \left(\sigma_x \tau_z \partial_x + \sigma_y \partial_y \right) \Psi$$

with Pauli matrices σ, τ corresponding to sublattice and valley

Add spin $\mathbf{s}$, obtain (for $\tau_z = 1$, i.e. around $\mathbf{K}$)

$$H_0(\mathbf{K} + \mathbf{k}) = \Psi^\dagger \begin{pmatrix} h_\uparrow(\mathbf{k}) & 0 \\ 0 & h_\downarrow(\mathbf{k}) \end{pmatrix} \Psi$$

Introduce TRI spin-orbit-coupling $H_{\mathrm{SO}} = \lambda_{\mathrm{SO}} \Psi^\dagger \sigma_z \tau_z s_z \Psi$

Doubly degenerate bands with a gap: $E_k = \pm\sqrt{(\hbar v_F k)^2 + \lambda_{\mathrm{SO}}}$

Tight binding model: two identical copies of the Haldane model

C. L. Kane, E. J. Mele. Phys. Rev. Lett., **95**, 226801 (2005)

$$H = t_1 \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{i\sigma} + i\lambda_{SO} \sum_{\langle\langle i,j \rangle\rangle, \sigma, \sigma'} \nu_{ij} c_{i\sigma}^\dagger s_{\sigma\sigma'}^z c_{j\sigma'}$$

The Kane and Mele model (II)

$$h_\uparrow(\mathbf{K}+\mathbf{k}) = \sigma_x k_x + \sigma_y k_y + \lambda_{SO}\sigma_z s_z \quad h_\uparrow(\mathbf{K}'+\mathbf{k}) = -\sigma_x k_x + \sigma_y k_y - \lambda_{SO}\sigma_z s_z$$
$$h_\downarrow(\mathbf{K}+\mathbf{k}) = \sigma_x k_x + \sigma_y k_y - \lambda_{SO}\sigma_z s_z \quad h_\downarrow(\mathbf{K}'+\mathbf{k}) = -\sigma_x k_x + \sigma_y k_y + \lambda_{SO}\sigma_z s_z$$

The Kane and Mele model (II)

$$h_{\uparrow}(\mathbf{K}+\mathbf{k}) = \sigma_x k_x + \sigma_y k_y + \lambda_{\mathrm{SO}} \sigma_z s_z \quad h_{\uparrow}(\mathbf{K}'+\mathbf{k}) = -\sigma_x k_x + \sigma_y k_y - \lambda_{\mathrm{SO}} \sigma_z s_z$$
$$h_{\downarrow}(\mathbf{K}+\mathbf{k}) = \sigma_x k_x + \sigma_y k_y - \lambda_{\mathrm{SO}} \sigma_z s_z \quad h_{\downarrow}(\mathbf{K}'+\mathbf{k}) = -\sigma_x k_x + \sigma_y k_y + \lambda_{\mathrm{SO}} \sigma_z s_z$$

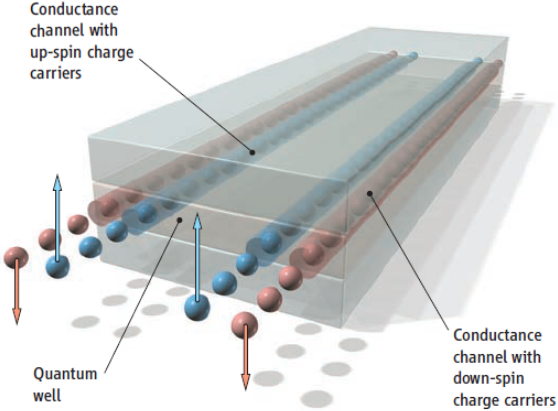
Time reversal symmetry $\mathcal{T}_1 = -i s_y K = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} K$

The Kane and Mele model (II)

$$h_\uparrow(\mathbf{K}+\mathbf{k}) = \sigma_x k_x + \sigma_y k_y + \lambda_{SO}\sigma_z s_z \quad h_\uparrow(\mathbf{K}'+\mathbf{k}) = -\sigma_x k_x + \sigma_y k_y - \lambda_{SO}\sigma_z s_z$$
$$h_\downarrow(\mathbf{K}+\mathbf{k}) = \sigma_x k_x + \sigma_y k_y - \lambda_{SO}\sigma_z s_z \quad h_\downarrow(\mathbf{K}'+\mathbf{k}) = -\sigma_x k_x + \sigma_y k_y + \lambda_{SO}\sigma_z s_z$$

Time reversal symmetry $\mathcal{T}_1 = -is_y K = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} K$

$h_{\uparrow(\downarrow)}$ known from Haldane model $\Rightarrow$ Hall conductance ± 1 for $\uparrow$ ($\downarrow$) = **quantum spin Hall effect**



Bernevig-Hughes-Zhang-model (I)

Apply construction rule on QWZ-model $\Rightarrow$ Bernevig-Hughes-Zhang-model

$$H_{\mathrm{BHZ}}(\mathbf{k}) = s_0 \otimes [(u + \cos k_x + \cos k_y)\,\sigma_z + \sin k_x \sigma_x] + s_z \otimes \sin k_y \sigma_y + s_x \otimes C$$

- 2d wavevector $\mathbf{k}$, staggered onsite potential u

Bernevig-Hughes-Zhang-model (I)

Apply construction rule on QWZ-model $\Rightarrow$ Bernevig-Hughes-Zhang-model

$$H_{\mathrm{BHZ}}(\mathbf{k}) = s_0 \otimes \left[\left(u + \cos k_x + \cos k_y \right) \sigma_z + \sin k_x \sigma_x \right] + s_z \otimes \sin k_y \sigma_y + s_x \otimes C$$

- 2d wavevector $\mathbf{k}$, staggered onsite potential u
- With $C = 0$ introduced as 4-band-model for HgTe
  B. A. Bernevig, T. L. Hughes, and S.-C. Zhang. Science, **314**, 1757 (2006)

Bernevig-Hughes-Zhang-model (I)

Apply construction rule on QWZ-model $\Rightarrow$ Bernevig-Hughes-Zhang-model

$$H_{\mathrm{BHZ}}(\mathbf{k}) = s_0 \otimes [(u + \cos k_x + \cos k_y)\,\sigma_z + \sin k_x \sigma_x] + s_z \otimes \sin k_y \sigma_y + s_x \otimes C$$

- 2d wavevector $\mathbf{k}$, staggered onsite potential u
- With $C = 0$ introduced as 4-band-model for HgTe
  B. A. Bernevig, T. L. Hughes, and S.-C. Zhang. Science, **314**, 1757 (2006)
- Two symmetries for $C = 0$: $\mathcal{T}_1\, H\, \mathcal{T}_1^{-1} = H = \mathcal{T}_2\, H\, \mathcal{T}_2^{-1}$

Bernevig-Hughes-Zhang-model (I)

Apply construction rule on QWZ-model $\Rightarrow$ Bernevig-Hughes-Zhang-model

$$H_{\mathrm{BHZ}}(\mathbf{k}) = s_0 \otimes [(u + \cos k_x + \cos k_y)\, \sigma_z + \sin k_x \sigma_x] + s_z \otimes \sin k_y \sigma_y + s_x \otimes C$$

- 2d wavevector $\mathbf{k}$, staggered onsite potential u
- With $C = 0$ introduced as 4-band-model for HgTe
  B. A. Bernevig, T. L. Hughes, and S.-C. Zhang. Science, **314**, 1757 (2006)
- Two symmetries for $C = 0$: $\mathcal{T}_1\, H\, \mathcal{T}_1^{-1} = H = \mathcal{T}_2\, H\, \mathcal{T}_2^{-1}$
- Introduce Hermitian coupling $C = C^\dagger$

Properties of edge states

$\mathbf{C = 0}$

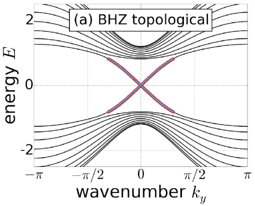
Direct sum of two Chern insulators with opposite Q

Kramers's pair of edge state branches on each edge

Edge states propagate in opposite directions

Edge state branches linked by both $\mathcal{T}_1$ & $\mathcal{T}_2$

TRI 1d Brillouin zone $-\pi \leq k_x \leq \pi$ has to be symmetric
$\Rightarrow N_+ = N_- \Leftrightarrow Q = N_+ - N_- = 0 = \text{const}$



Parameter $u = 1.2$

Properties of edge states

$\mathbf{C} = \mathbf{0}$

Direct sum of two Chern insulators with opposite Q

Kramers's pair of edge state branches on each edge

Edge states propagate in opposite directions

Edge state branches linked by both $\mathcal{T}_1$ & $\mathcal{T}_2$

TRI 1d Brillouin zone $-\pi \leq k_x \leq \pi$ has to be symmetric
$\Rightarrow N_+ = N_- \Leftrightarrow Q = N_+ - N_- = 0 = \text{const}$

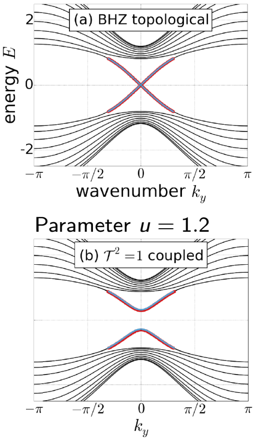
$\mathbf{C} = \mathbf{C^T}$

Example with $C = 0.3\sigma_x$

Respects $\mathcal{T}^2 = 1$ symmetry

Can gap edge state branches out

Hopping between counterpropagating edge states



Properties of edge states

$\mathbf{C = 0}$

Direct sum of two Chern insulators with opposite Q

Kramers's pair of edge state branches on each edge

Edge states propagate in opposite directions

Edge state branches linked by both $\mathcal{T}_1$ & $\mathcal{T}_2$

TRI 1d Brillouin zone $-\pi \leq k_x \leq \pi$ has to be symmetric
$\Rightarrow N_+ = N_- \Leftrightarrow Q = N_+ - N_- = 0 = \text{const}$

$\mathbf{C = C^T}$

Example with $C = 0.3\sigma_x$

Respects $\mathcal{T}^2 = 1$ symmetry

Can gap edge state branches out

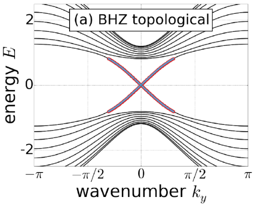
Hopping between counterpropagating edge states

$\mathbf{C = -C^T}$

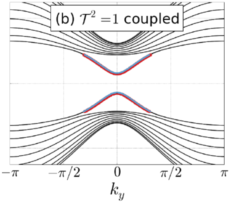
Example with $C = 0.3\sigma_y$

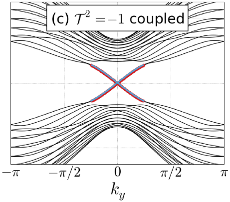
Respects $\mathcal{T}^2 = -1$ symmetry

Crossing is protected



Parameter $u = 1.2$



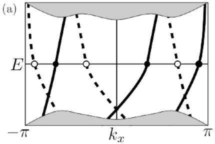


Edge states in $\mathcal{T}^2 = -1$

$Q = 0$ & symmetry of the BZ $\Rightarrow$ creation/annihilation of egde states in pairs of Kramer's pairs

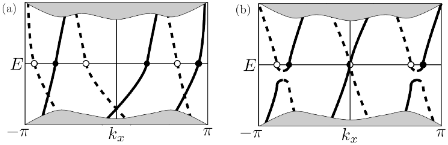
Edge states in $\mathcal{T}^2 = -1$

$Q = 0$ & symmetry of the BZ $\Rightarrow$ creation/annihilation of egde states in pairs of Kramer's pairs



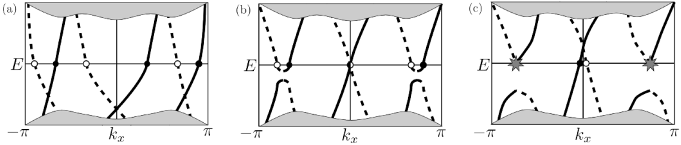
Edge states in $\mathcal{T}^2 = -1$

$Q = 0$ & symmetry of the BZ $\Rightarrow$ creation/annihilation of egde states in pairs of Kramer's pairs



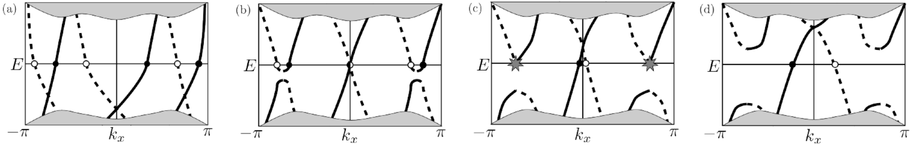
Edge states in $\mathcal{T}^2 = -1$

$Q = 0$ & symmetry of the BZ $\Rightarrow$ creation/annihilation of egde states in pairs of Kramer's pairs



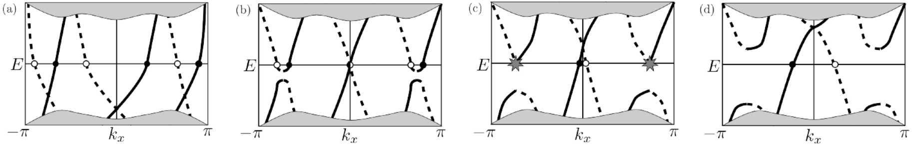
Edge states in $\mathcal{T}^2 = -1$

$Q = 0$ & symmetry of the BZ $\Rightarrow$ creation/annihilation of egde states in pairs of Kramer's pairs



Edge states in $\mathcal{T}^2 = -1$

$Q = 0$ & symmetry of the BZ $\Rightarrow$ creation/annihilation of egde states in pairs of Kramer's pairs



Crossings at $k_x = 0, \pm\pi$ (TRIM) are protected by Kramer's degeneracy

The $\mathbb{Z}_2$-invariant

Parity of the number of egde states is well defined at any E

Parity of the number of egde states is (almost) independent of E

The $\mathbb{Z}_2$-invariant

Parity of the number of egde states is well defined at any E

Parity of the number of egde states is (almost) independent of E

Number of egde states changes only by $4 \cdot n$, $n \in \mathbb{N}$, for $\mathcal{T}^2 = -1$

$\Rightarrow$ Parity of $N(E) = N_+(E) + N_-(E)$ conserved:

$$D(E) = \frac{N(E)}{2} \mod 2 = \text{const} \in \{0, 1\}$$

Topological invariant for TRI 2d lattices

The $\mathbb{Z}_2$-invariant

Parity of the number of egde states is well defined at any E

Parity of the number of egde states is (almost) independent of E

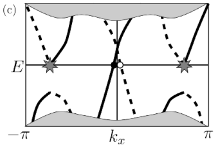
Number of egde states changes only by $4 \cdot n$, $n \in \mathbb{N}$, for $\mathcal{T}^2 = -1$

$\Rightarrow$ Parity of $N(E) = N_+(E) + N_-(E)$ conserved:

$$D(E) = \frac{N(E)}{2} \mod 2 = \text{const} \in \{0, 1\}$$

Topological invariant for TRI 2d lattices

Not well defined on sets of zero measure:



Absence of backscattering: the scattering matrix (I)

Absence of backscattering: the scattering matrix (I)

Consider a 2d lattice with internal degree of freedom

Finite extent in y-direction and periodic boundaries in x

rectangular unit cells $N_x \times N_y$ and translational invariance along x

Absence of backscattering: the scattering matrix (I)

Consider a 2d lattice with internal degree of freedom

Finite extent in y-direction and periodic boundaries in x

rectangular unit cells $N_x \times N_y$ and translational invariance along x

States from Bloch theorem:	$\|l, \pm\rangle = \|k_{l,\pm}\rangle \otimes \|\phi_{l,\pm}\rangle$
left / right propagating ($\pm$)	with mode $l \in \{1, ..., N\}$
usual momentum eigenstates:	$\|k_{l,\pm}\rangle = \frac{1}{\sqrt{N_x}} \sum_{m_x=1}^{N_x} e^{ikm_x} \|m_x\rangle$
transverse mode, inernal DOF:	$\|\phi_{l,\pm}\rangle$
with group velocity:	$v_{l,\pm} = \left.\frac{\partial E(k)}{\partial k}\right\|_{k=k_{l,\pm}}$

Absence of backscattering: the scattering matrix (I)

Consider a 2d lattice with internal degree of freedom

Finite extent in y-direction and periodic boundaries in x

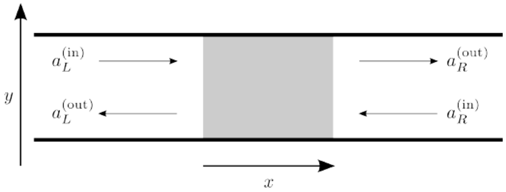
rectangular unit cells $N_x \times N_y$ and translational invariance along x

States from Bloch theorem:	$\vert l, \pm\rangle = \vert k_{l,\pm}\rangle \otimes \vert\phi_{l,\pm}\rangle$
left / right propagating ($\pm$)	with mode $l \in \{1, ..., N\}$
usual momentum eigenstates:	$\vert k_{l,\pm}\rangle = \frac{1}{\sqrt{N_x}} \sum_{m_x=1}^{N_x} e^{ikm_x} \vert m_x\rangle$
transverse mode, inernal DOF:	$\vert\phi_{l,\pm}\rangle$
with group velocity:	$v_{l,\pm} = \frac{\partial E(k)}{\partial k}\Big\vert_{k=k_{l,\pm}}$

Re-normalize: different states carry same particle current trough arbitrary cross section

$$|l, \pm\rangle_c = \frac{1}{\sqrt{|v_{l,\pm}|}} |l, \pm\rangle$$

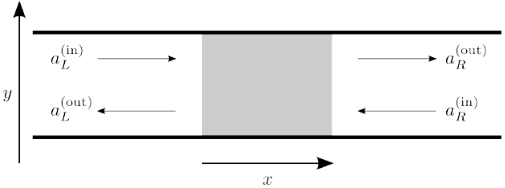
Absence of backscattering: the scattering matrix (II)



Consider scattering at disordered region (gray) with respect to coefficients:

$$\mathbf{a}^{\mathrm{in}} = \left(a^{\mathrm{in}}_{L,1}, ...a^{\mathrm{in}}_{L,N}, a^{\mathrm{in}}_{R,1}, ..., a^{\mathrm{in}}_{R,N}\right) \mapsto \mathbf{a}^{\mathrm{out}} = \left(a^{\mathrm{out}}_{L,1}, ...a^{\mathrm{out}}_{L,N}, a^{\mathrm{out}}_{R,1}, ..., a^{\mathrm{out}}_{R,N}\right)$$

Absence of backscattering: the scattering matrix (II)



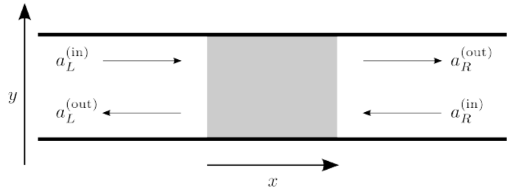
Consider scattering at disordered region (gray) with respect to coefficients:

$$\mathbf{a}^{\mathrm{in}} = \left(a_{L,1}^{\mathrm{in}}, \ldots a_{L,N}^{\mathrm{in}}, a_{R,1}^{\mathrm{in}}, \ldots, a_{R,N}^{\mathrm{in}}\right) \mapsto \mathbf{a}^{\mathrm{out}} = \left(a_{L,1}^{\mathrm{out}}, \ldots a_{L,N}^{\mathrm{out}}, a_{R,1}^{\mathrm{out}}, \ldots, a_{R,N}^{\mathrm{out}}\right)$$

Energy eigenstate outside scattering region:

$$|\psi\rangle = \sum_{l=1}^{N} a_{L,l}^{\mathrm{in}} \,|l,+,L\rangle_c + a_{R,l}^{\mathrm{in}} \,|l,-,R\rangle_c + a_{L,l}^{\mathrm{out}} \,|l,-,L\rangle_c + a_{R,l}^{\mathrm{out}} \,|l,+,R\rangle_c$$

Absence of backscattering: the scattering matrix (II)



Consider scattering at disordered region (gray) with respect to coefficients:

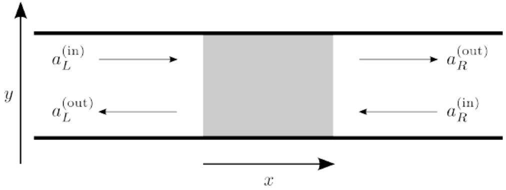
$$\mathbf{a}^{\text{in}} = \left(a^{\text{in}}_{L,1}, \ldots a^{\text{in}}_{L,N}, a^{\text{in}}_{R,1}, \ldots, a^{\text{in}}_{R,N}\right) \mapsto \mathbf{a}^{\text{out}} = \left(a^{\text{out}}_{L,1}, \ldots a^{\text{out}}_{L,N}, a^{\text{out}}_{R,1}, \ldots, a^{\text{out}}_{R,N}\right)$$

Energy eigenstate outside scattering region:

$$|\psi\rangle = \sum_{l=1}^{N} a^{\text{in}}_{L,l} \, |l, +, L\rangle_c + a^{\text{in}}_{R,l} \, |l, -, R\rangle_c + a^{\text{out}}_{L,l} \, |l, -, L\rangle_c + a^{\text{out}}_{R,l} \, |l, +, R\rangle_c$$

Unitary $2N \times 2N$ scattering matrix S: $\mathbf{a}^{\text{out}} = S \, \mathbf{a}^{\text{in}} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \mathbf{a}^{\text{in}}$

Absence of backscattering: the scattering matrix (II)



Consider scattering at disordered region (gray) with respect to coefficients:

$$\mathbf{a}^{\mathrm{in}} = \left(a^{\mathrm{in}}_{L,1}, \ldots a^{\mathrm{in}}_{L,N}, a^{\mathrm{in}}_{R,1}, \ldots, a^{\mathrm{in}}_{R,N}\right) \mapsto \mathbf{a}^{\mathrm{out}} = \left(a^{\mathrm{out}}_{L,1}, \ldots a^{\mathrm{out}}_{L,N}, a^{\mathrm{out}}_{R,1}, \ldots, a^{\mathrm{out}}_{R,N}\right)$$

Energy eigenstate outside scattering region:

$$|\psi\rangle = \sum_{l=1}^{N} a^{\mathrm{in}}_{L,l} \, |l,+,L\rangle_c + a^{\mathrm{in}}_{R,l} \, |l,-,R\rangle_c + a^{\mathrm{out}}_{L,l} \, |l,-,L\rangle_c + a^{\mathrm{out}}_{R,l} \, |l,+,R\rangle_c$$

Unitary $2N \times 2N$ scattering matrix S: $\mathbf{a}^{\mathrm{out}} = S \, \mathbf{a}^{\mathrm{in}} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \mathbf{a}^{\mathrm{in}}$

Eigenvalues of $tt^\dagger$, $t't'^\dagger$, $1 - rr^\dagger$ and $1 - r'r'^\dagger$ are the same
(*transmission eigenvalues*)

Absence of backscattering: scattering of Kramer's pairs of edge states (I)

Choose propagating modes such that $\mathcal{T}|l,-,L\rangle_c = |l,+,L\rangle_c$

$$\mathcal{T}|l,+,R\rangle_c = |l,-,R\rangle_c$$

Absence of backscattering: scattering of Kramer's pairs of edge states (I)

Choose propagating modes such that $\mathcal{T}|l,-,L\rangle_c = |l,+,L\rangle_c$
$$\mathcal{T}|l,+,R\rangle_c = |l,-,R\rangle_c$$

Energy eigenstate outside scattering region:

$$|\psi\rangle = \sum_{l=1}^{N} a^{\text{in}}_{L,l}\,|l,+,L\rangle_c + a^{\text{in}}_{R,l}\,|l,-,R\rangle_c + \left(S\,a^{\text{in}}\right)_{L,l}\,|l,-,L\rangle_c + \left(S\,a^{\text{in}}\right)_{R,l}\,|l,+,R\rangle_c$$

Absence of backscattering: scattering of Kramer's pairs of edge states (I)

Choose propagating modes such that $\mathcal{T}|l,-,L\rangle_c = |l,+,L\rangle_c$

$$\mathcal{T}|l,+,R\rangle_c = |l,-,R\rangle_c$$

Energy eigenstate outside scattering region:

$$|\psi\rangle = \sum_{l=1}^{N} a^{\text{in}}_{L,l}\,|l,+,L\rangle_c + a^{\text{in}}_{R,l}\,|l,-,R\rangle_c + \left(S\,a^{\text{in}}\right)_{L,l}\,|l,-,L\rangle_c + \left(S\,a^{\text{in}}\right)_{R,l}\,|l,+,R\rangle_c$$

Given a TRI scatterer & using $\mathcal{T}^2 = -1$, another eigenstate at same energy is

$$\begin{aligned}
-\mathcal{T}|\psi\rangle &= \sum_{l=1}^{N} -a^{\text{in}*}_{L,l}\,|l,-,L\rangle_c - a^{\text{in}*}_{R,l}\,|l,+,R\rangle_c + \left(S^*\,a^{\text{in}*}\right)_{L,l}\,|l,+,L\rangle_c + \left(S^*\,a^{\text{in}*}\right)_{R,l}\,|l,-,R\rangle_c \\
&= \sum_{l=1}^{N} \left(S^* a^{\text{in}*}\right)_{L,l}\,|l,+,L\rangle_c + \left(S^* a^{\text{in}*}\right)_{R,l}\,|l,-,R\rangle_c \\
&\qquad + \left(-S^T S^*\,a^{\text{in}*}\right)_{L,l}\,|l,-,L\rangle_c + \left(-S^T S^*\,a^{\text{in}*}\right)_{R,l}\,|l,+,R\rangle_c
\end{aligned}$$

Absence of backscattering: scattering of Kramer's pairs of edge states (I)

Choose propagating modes such that $\mathcal{T}|l,-,L\rangle_c = |l,+,L\rangle_c$

$$\mathcal{T}|l,+,R\rangle_c = |l,-,R\rangle_c$$

Energy eigenstate outside scattering region:

$$|\psi\rangle = \sum_{l=1}^{N} a^{\text{in}}_{L,l}\,|l,+,L\rangle_c + a^{\text{in}}_{R,l}\,|l,-,R\rangle_c + \left(S\,a^{\text{in}}\right)_{L,l}\,|l,-,L\rangle_c + \left(S\,a^{\text{in}}\right)_{R,l}\,|l,+,R\rangle_c$$

Given a TRI scatterer & using $\mathcal{T}^2 = -1$, another eigenstate at same energy is

$$-\mathcal{T}|\psi\rangle = \sum_{l=1}^{N} -a^{\text{in}*}_{L,l}\,|l,-,L\rangle_c - a^{\text{in}*}_{R,l}\,|l,+,R\rangle_c + \left(S^*\,a^{\text{in}*}\right)_{L,l}\,|l,+,L\rangle_c + \left(S^*\,a^{\text{in}*}\right)_{R,l}\,|l,-,R\rangle_c$$

$$= \sum_{l=1}^{N} \left(S^* a^{\text{in}*}\right)_{L,l}\,|l,+,L\rangle_c + \left(S^* a^{\text{in}*}\right)_{R,l}\,|l,-,R\rangle_c$$

$$+ \left(-S^T S^*\,a^{\text{in}*}\right)_{L,l}\,|l,-,L\rangle_c + \left(-S^T S^*\,a^{\text{in}*}\right)_{R,l}\,|l,+,R\rangle_c$$

S is uniquely at any energy $\Rightarrow S = -S^T$

Absence of backscattering: scattering of Kramer's pairs of edge states (II)

With $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ follows $r = -r^T$ and $r' = -r'^T$

Absence of backscattering: scattering of Kramer's pairs of edge states (II)

With $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ follows $r = -r^T$ and $r' = -r'^T$

This implies $\det(r) = \det(r^T) = \det(-r) = (-1)^N \det(r)$

For an odd number N of Kramer's pairs: $\det(r) = \det(r') = 0$

$\Rightarrow \det(rr^\dagger) = \det(r)\det(r^\dagger) = 0$ and the same for r'

Absence of backscattering: scattering of Kramer's pairs of edge states (II)

With $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ follows $r = -r^T$ and $r' = -r'^T$

This implies $\det(r) = \det(r^T) = \det(-r) = (-1)^N \det(r)$

For an odd number N of Kramer's pairs: $\det(r) = \det(r') = 0$

$\Rightarrow \det(rr^\dagger) = \det(r)\det(r^\dagger) = 0$ and the same for r'

For $D = 1$ at least one transmission eigenvalue is 1, i.e. there is at least one linear combination of incoming edge states from both sides that is perfectly transmitted through a time-reversal symmetryc defect.

Absence of backscattering: scattering of Kramer's pairs of edge states (II)

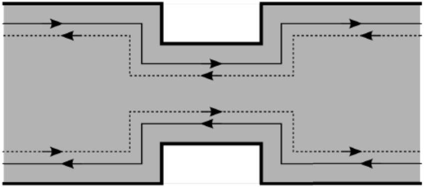
With $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ follows $r = -r^T$ and $r' = -r'^T$

This implies $\det(r) = \det(r^T) = \det(-r) = (-1)^N \det(r)$

For an odd number N of Kramer's pairs: $\det(r) = \det(r') = 0$

$\Rightarrow \det(rr^\dagger) = \det(r)\det(r^\dagger) = 0$ and the same for r'

For $D = 1$ at least one transmission eigenvalue is 1, i.e. there is at least one linear combination of incoming edge states from both sides that is perfectly transmitted through a time-reversal symmetryc defect.



No backscattering at small constriction

Absence of backscattering: scattering of Kramer's pairs of edge states (II)

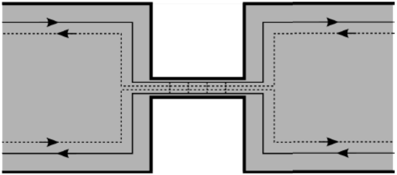
With $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ follows $r = -r^T$ and $r' = -r'^T$

This implies $\det(r) = \det(r^T) = \det(-r) = (-1)^N \det(r)$

For an odd number N of Kramer's pairs: $\det(r) = \det(r') = 0$

$\Rightarrow \det(rr^\dagger) = \det(r)\det(r^\dagger) = 0$ and the same for r'

For $D = 1$ at least one transmission eigenvalue is 1, i.e. there is at least one linear combination of incoming edge states from both sides that is perfectly transmitted through a time-reversal symmetryc defect.



Constriction of the order of the edge states penetration depth:
backscattering between edges possible

Absence of backscattering: scattering of Kramer's pairs of edge states (II)

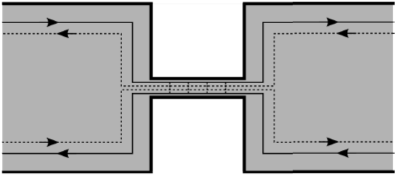
With $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ follows $r = -r^T$ and $r' = -r'^T$

This implies $\det(r) = \det(r^T) = \det(-r) = (-1)^N \det(r)$

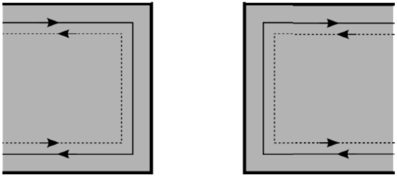
For an odd number N of Kramer's pairs: $\det(r) = \det(r') = 0$

$\Rightarrow \det(rr^\dagger) = \det(r)\det(r^\dagger) = 0$ and the same for r'

For $D = 1$ at least one transmission eigenvalue is 1, i.e. there is at least one linear combination of incoming edge states from both sides that is perfectly transmitted through a time-reversal symmetryc defect.



Constriction of the order of the edge states penetration depth:
backscattering between edges possible



Zero transmission = backscattering

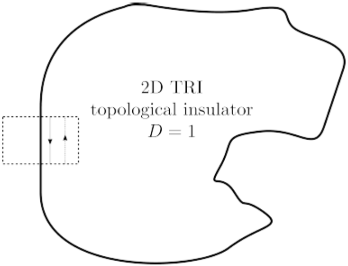
Absence of backscattering: robustness against disorder

Finite disordered TRI 2d topological insulator with arbitrary geometry and TRS disorder

Inside the box switch disorder off and straighten geometry out

Apply adiabatic deformation in the box make such that there is only one Kramer's pair of edge states

Disordered region is time reversal symmetric scatterer



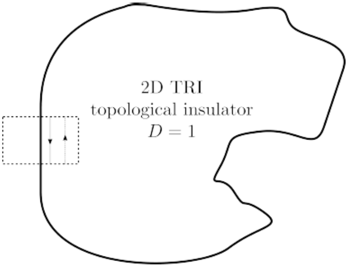
Absence of backscattering: robustness against disorder

Finite disordered TRI 2d topological insulator with arbitrary geometry and TRS disorder

Inside the box switch disorder off and straighten geometry out

Apply adiabatic deformation in the box make such that there is only one Kramer's pair of edge states

Disordered region is time reversal symmetric scatterer



$\Rightarrow$ Any segment of the edge supports at least one perfectly transmitted Kramer's pair of edge states

Experimental realisation (I)

Conventional semiconductor: the *p*-orbital-like band is below the *s*-like band

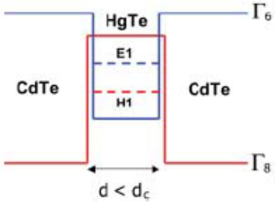
In HgTe these bands are inverted.

Experimental realisation (I)

Conventional semiconductor: the *p*-orbital-like band is below the *s*-like band

In HgTe these bands are inverted.

Construct quantum wells of HgTe-layers in CdTe:



Experimental realisation (I)

Conventional semiconductor: the *p*-orbital-like band is below the *s*-like band

In HgTe these bands are inverted.

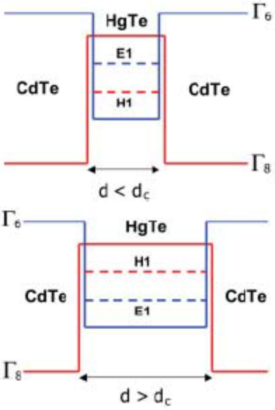
Construct quantum wells of HgTe-layers in CdTe:

At thickness $d_c \approx 64\,\text{Å}$ the bands do cross

Theory predicts $D = 1$ for $d > d_c$

B. A. Bernevig, T. L. Hughes, and S.-C. Zhang. Science, **314**, 1757 (2006)

Expect quantum spin Hall effect

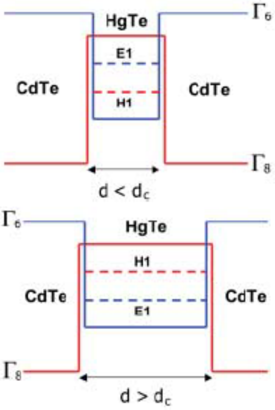


Experimental realisation (I)

Conventional semiconductor: the *p*-orbital-like band is below the *s*-like band

In HgTe these bands are inverted.

Construct quantum wells of HgTe-layers in CdTe:

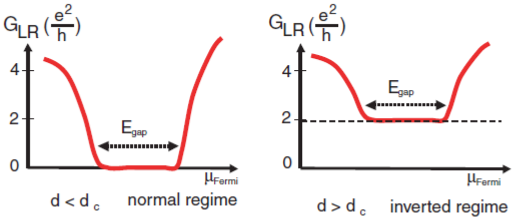


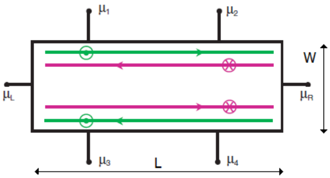
At thickness $d_c \approx 64\,\text{Å}$ the bands do cross

Theory predicts $D = 1$ for $d > d_c$

B. A. Bernevig, T. L. Hughes, and S.-C. Zhang. Science, **314**, 1757 (2006)

Expect quantum spin Hall effect





Experimental realisation (II)

Six-terminal measurements with HgTe quantum wells of different size

M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, S.-C. Zhang. Science, **318**, 766 (2007)

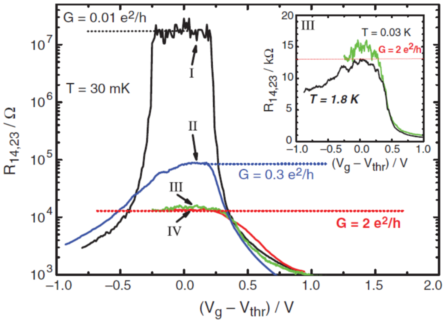
Confirmed phase transition at $d_c = 63(1)\,\text{Å}$

Experimental realisation (II)

Six-terminal measurements with HgTe quantum wells of different size

M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, S.-C. Zhang. Science, **318**, 766 (2007)

Confirmed phase transition at $d_c = 63(1)\,\text{Å}$



	d / Å	$L \times W$ / μm^2
I	55	20.0×13.3
II	73	20.0×13.3
III	73	1.0×1.0
IV	73	1.0×0.5

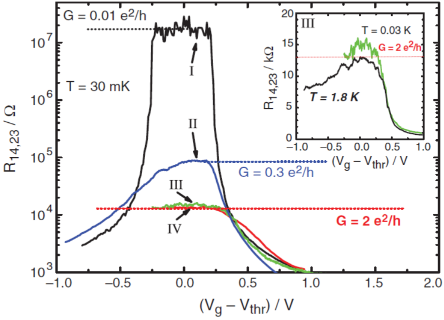
Insert: both samples like III

Experimental realisation (II)

Six-terminal measurements with HgTe quantum wells of different size

M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, S.-C. Zhang. Science, **318**, 766 (2007)

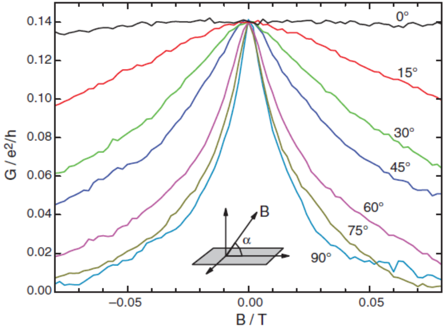
Confirmed phase transition at $d_c = 63(1)$ Å



	d / Å	$L \times W$ / μm^2
I	55	20.0×13.3
II	73	20.0×13.3
III	73	1.0×1.0
IV	73	1.0×0.5

Insert: both samples like III

Breaking time reversal symmetry?
Sample like II



Literature

[1] J. K. Asbóth, L. Oroszlány, A. Pályi: *A Short Course on Topological Insulators*. `https://arxiv.org/abs/1509.02295`, September 9, 2015

[2] B. A. Bernevig, T. L. Hughes: *Topological insulators and topological superconductors*. Princeton University Press, Princeton (2013)

[3] C. L. Kane, E. J. Mele: *Quantum Spin Hall Effect in Graphene*. Phys. Rev. Lett., **95**, 226801 (2005)

[4] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang: *Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells*. Science, **314**, 1757 (2006)

[5] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, S.-C. Zhang. *Quantum Spin Hall Insulator State in HgTe Quantum Wells*. Science, **318**, 766 (2007)