Non-Abelian Berry phase and topological spin-currents

Clara Mühlherr

University of Constance

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Non-degenerate levels

Schrödinger equation

$$\hat{H}(\boldsymbol{\theta})\,|\Psi_n(\boldsymbol{\theta})\rangle = E_n\,|\Psi_n(\boldsymbol{\theta})\rangle$$

$\boldsymbol{\theta} = (\theta^1, \ldots, \theta^r)$: set of parameters
non-degenerate state $|\Psi_n(\boldsymbol{\theta})\rangle$ for any $\boldsymbol{\theta}$

- Berry connection:

$$\boldsymbol{A}_n = i\,\langle\Psi_n(\boldsymbol{\theta})|\partial_{\boldsymbol{\theta}}\Psi(\boldsymbol{\theta})\rangle$$

- Berry phase along $\mathcal{C}$:

$$\gamma_n(\mathcal{C}) = \oint_{\mathcal{C}} \boldsymbol{A}_n(\boldsymbol{\theta})\cdot \mathrm{d}\boldsymbol{\theta}$$

$\gamma_n(\mathcal{C})$ gauge invariant, but

$$|\Psi_n(\boldsymbol{\theta})\rangle \to e^{i\alpha(\boldsymbol{\theta})}\,|\Psi_n(\boldsymbol{\theta})\rangle : \qquad \boldsymbol{A}_n(\boldsymbol{\theta}) \to \boldsymbol{A}_n(\boldsymbol{\theta}) + \partial_{\boldsymbol{\theta}}\alpha(\boldsymbol{\theta})$$

Non-degenerate levels

- Berry curvature:

$$\oint_{\partial\mathcal{F}} \boldsymbol{A}_n(\boldsymbol{\theta}) \cdot \mathrm{d}\boldsymbol{\theta} = \int_{\mathcal{F}} B_n(\boldsymbol{\theta}) \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^r$$

$B_n(\boldsymbol{\theta})$ gauge invariant

- explicit form:

$$B^{(n)}_{\mu\nu}(\boldsymbol{\theta}) = \frac{\partial}{\partial\theta^\mu} A^{(n)}_\nu(\boldsymbol{\theta}) - \frac{\partial}{\partial\theta^\nu} A^{(n)}_\mu(\boldsymbol{\theta})$$

- holds if

$$\left[A^{(n)}_\mu(\boldsymbol{\theta}), A^{(n)}_\nu(\boldsymbol{\theta})\right] = 0$$

$\rightarrow$ for Abelian gauge fields, i.e. non-degenerate levels

Occurrence of degeneracies

In physics we can not avoid degeneracies, for example: Kramer's theorem

We have to deal with degenerate states

$\Rightarrow$ non-Abelian gauge theory

Gauge fields in presence of degeneracies

time-dependent Schrödinger equation

$$H(\boldsymbol{\theta}(t))\,|\Psi_n(t)\rangle = i\frac{\partial}{\partial t}\,|\Psi_n(t)\rangle$$

- m_n degenerate sublevels in subspace $\mathcal{H}_n$
- $\boldsymbol{\theta}(t)$ set of parameters, varied in time $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_i \;\rightarrow\; \boldsymbol{\theta}(T) = \boldsymbol{\theta}_f$
- adiabatic theorem:
  By varying $\boldsymbol{\theta}(t)$, subspaces do not cross each other
  $|\Psi_{n,m}(0)\rangle \in \mathcal{H}_n$ are mapped to $|\Psi_{n,m'}(T)\rangle \in \mathcal{H}_n$

Gauge fields in presence of degeneracies

- $|\Psi_a(t)\rangle$ set of basis functions of subspace $\mathcal{H}_n$

$$\text{choose locally} \qquad H(\boldsymbol{\theta}(t))\,|\Psi_a(t)\rangle = 0$$

- unitary $U(t)$ maps those solutions to functions $|\Phi_a(t)\rangle \in \mathcal{H}_n$

$$|\Phi_a(t)\rangle = U_{ab}(t)\,|\Psi_b(t)\rangle\,, \qquad |\Phi_a(0)\rangle \stackrel{!}{=} |\Psi_a(0)\rangle$$

- $|\Phi_a(t)\rangle$ remain normalized

$$\Rightarrow \ \langle\Phi_b|\partial_t\Phi_a\rangle = 0 \ \Leftrightarrow \ (U^{-1}\partial_t U)_{ba} = \langle\Psi_b|\partial_t\Psi_a\rangle \equiv A_{ab}$$

$A(t)$: gauge potential, depends on geometry of subspace n

Gauge fields in presence of degeneracies

Gauge potential

$$A_{ab} = (U^{-1}\partial_t U)_{ba}$$

$\Rightarrow$ $U(t) = \mathcal{P}\exp \oint A(\tau)\mathrm{d}\tau$

- arbitrary transformation $R(t) \in SO(m_n)$:

$$\begin{aligned} |\Psi_a(t)\rangle &\rightarrow R(t)\,|\Psi_a(t)\rangle \\ A(t) &\rightarrow \partial_t R(t)R^{-1}(t) + R(t)A(t)R^{-1}(t) \\ U(t) &\rightarrow R(t)U(t)R^{-1}(t) \end{aligned}$$

$\Rightarrow$ eigenvalues of mapping $U : \mathcal{H}_n \rightarrow \mathcal{H}_n$ between degenerate sublevels are gauge invariant, i.e. potentially observable

- $m_n = 1$: $U(t) = e^{-i\Delta\gamma}$ adds a simple phase factor

Gauge fields in presence of degeneracies

Generalize gauge potential $A(t)$ to parameter space M, where $\dim(M) = r$:

$$\boldsymbol{\theta} = (\theta^1, \ldots, \theta^r)$$

$\Rightarrow$ $A^T_\mu = \langle\Psi|\frac{\partial}{\partial\theta^\mu}\rangle = \langle\Psi|\partial_\mu|\Psi\rangle$

- Loop in parameter space ($\boldsymbol{\theta}_i = \boldsymbol{\theta}_f$)

$\Rightarrow$ mapping between degenerate sates: Wilson loop

$$U = \mathcal{P}\exp\oint \boldsymbol{A}_n \cdot \mathrm{d}\boldsymbol{\theta}$$

- non-degenerate levels ($m_n = 1$): $U \rightarrow$ Berry phase

Gauge fields in presence of degeneracies

What is the curvature of $\boldsymbol{A}$ in case of existing degeneracies?

- non-degenerate levels:

$$B^{(n)}_{\mu\nu}(\boldsymbol{\theta}) = \frac{\partial}{\partial\theta^\mu} A^{(n)}_\nu(\boldsymbol{\theta}) - \frac{\partial}{\partial\theta^\nu} A^{(n)}_\mu(\boldsymbol{\theta})$$

- more general definition:

$$B_{\mu\nu} = -\left[D_\mu, D_\nu\right]$$

$D_\mu = \partial_\mu - A_\mu$: covariant derivative ($D_\mu \ \rightarrow \ R(\boldsymbol{\theta}) D_\mu$)

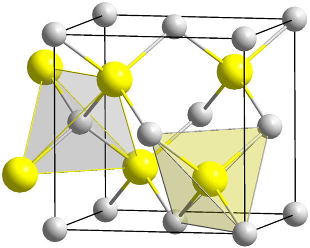
$$\begin{aligned} \Rightarrow \quad B_{\mu\nu} &= -\left[D_\mu, D_\nu\right] \\ &= -[\partial_\mu - A_\mu, \partial_\nu - A_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \end{aligned}$$

Degeneracies in real insulators

- Consider Si, Ge, GaAs, InSb
- Top of VB: $P_{3/2}$ levels
- diamond structure, inversion symmetry, rotational symmetry

$\Rightarrow$ terms in Hamiltonian: $\boldsymbol{k}^2$ and $(\boldsymbol{k} \cdot \boldsymbol{S})^2$

$\Rightarrow$ effective Luttinger Hamiltonian for holes

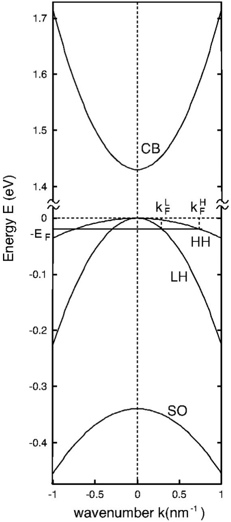


de.wikipedia.org/wiki/Galliumarsenid

$$H_0 = \frac{\hbar^2}{2m}\left((\gamma_1 + \frac{5}{2}\gamma_2)k^2 - 2\gamma_2(\boldsymbol{k} \cdot \boldsymbol{S})^2\right) = H_0(\boldsymbol{k})$$

$\gamma_{1/2}$: Luttinger parameters

Dispersion relation



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$$H_0(\boldsymbol{k}) = \frac{\hbar^2}{2m}\left((\gamma_1 + \frac{5}{2}\gamma_2)k^2 - 2\gamma_2(\boldsymbol{k}\cdot\boldsymbol{S})^2\right)$$

- diagonal in helicity basis ($\lambda = \hat{\boldsymbol{k}}\cdot\boldsymbol{S}$)
- eigenenergies

$$E_\lambda(\boldsymbol{k}) = \frac{\hbar^2 k^2}{2m}\left(\gamma_1 + \left(\frac{5}{2} - 2\lambda^2\right)\gamma_2\right) \equiv \frac{\hbar^2 k^2}{2m_\lambda}$$

$P_{3/2} \to S = \pm\frac{1}{2}, \pm\frac{3}{2} \to \lambda = \pm\frac{1}{2}, \pm\frac{3}{2}$

$\Rightarrow$ light and heavy holes:
LH and HH band
4-fold degeneracy lifted for $k \neq 0$

Effect of electric field

Introduce potential $V(\boldsymbol{x}) = e\boldsymbol{E} \cdot \boldsymbol{x}$

$$\Rightarrow \quad H = H_0 + V(\boldsymbol{x})$$

Diagonalization of $H_0(\boldsymbol{k})$, $\boldsymbol{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

- rotate by $\quad R(\boldsymbol{k}) = \exp(i\theta S_y)\exp(i\phi S_x)$

$$\Rightarrow \qquad R(\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{S})R^\dagger(\boldsymbol{k}) = kS_z$$

$$\Rightarrow \quad \tilde{H} = \frac{\hbar^2 k^2}{2m}\left(\gamma_1 + \frac{5}{2}\gamma_2 - 2\gamma_2 S_z^2\right) + R(\boldsymbol{k})V(\boldsymbol{x})R^\dagger(\boldsymbol{k})$$

- $\boldsymbol{x} = i\partial_{\boldsymbol{k}}$
  $\rightarrow$ transformation gives $V(\tilde{\boldsymbol{D}})$

$$\tilde{\boldsymbol{D}} = i\partial_{\boldsymbol{k}} + iR(\boldsymbol{k})\partial_{\boldsymbol{k}}R^\dagger(\boldsymbol{k}) = i\partial_{\boldsymbol{k}} - \tilde{\boldsymbol{A}}$$

$\tilde{\boldsymbol{A}}$: pure gauge potential $(F_{ij} = i[D_i, D_j] = 0)$

Pure gauge potential $\tilde{\boldsymbol{A}}$

Spin 3/2-matrices

$$S_x = \begin{pmatrix} & \frac{\sqrt{3}}{2} & & \\ \frac{\sqrt{3}}{2} & & 1 & \\ & 1 & & \frac{\sqrt{3}}{2} \\ & & \frac{\sqrt{3}}{2} & \end{pmatrix}, S_y = \begin{pmatrix} & \frac{-i\sqrt{3}}{2} & & \\ i\frac{\sqrt{3}}{2} & & -i & \\ & i & & -i\frac{\sqrt{3}}{2} \\ & & \frac{\sqrt{3}}{2} & \end{pmatrix}, S_z = \begin{pmatrix} \frac{3}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{3}{2} \end{pmatrix}$$

$$\tilde{\boldsymbol{A}} \cdot \mathrm{d}\boldsymbol{k} = \begin{pmatrix} -\frac{3}{2}\cos\theta\ \mathrm{d}\varphi & \frac{\sqrt{3}}{2}(\sin\theta\ \mathrm{d}\varphi + i\ \mathrm{d}\theta) & & \\ \frac{\sqrt{3}}{2}(\sin\theta\ \mathrm{d}\varphi - i\ \mathrm{d}\theta) & -\frac{1}{2}\cos\theta\ \mathrm{d}\varphi & \sin\theta\ \mathrm{d}\varphi + i\ \mathrm{d}\theta & \\ & \sin\theta\ \mathrm{d}\varphi - i\ \mathrm{d}\theta & \frac{1}{2}\cos\theta\ \mathrm{d}\varphi & \frac{\sqrt{3}}{2}(\sin\theta\ \mathrm{d}\varphi + i\ \mathrm{d}\theta) \\ & & \frac{\sqrt{3}}{2}(\sin\theta\ \mathrm{d}\varphi - i\ \mathrm{d}\theta) & \frac{3}{2}\cos\theta\ \mathrm{d}\varphi \end{pmatrix}$$

Approximations

Adiabatic Approximation: Neglect interband transitions (off-block-diagonal matrix elements of $\tilde{\boldsymbol{A}}$)

$$\Rightarrow \boldsymbol{A}' \cdot \mathrm{d}\boldsymbol{k} =$$

- existing degeneracies $\to$ $\boldsymbol{A}'$ non-Abelian

Abelian approximation: neglect off-diagonal elements

$$\Rightarrow \boldsymbol{A}'_{\mathrm{A}} = -S_z \cos\theta \; \mathrm{d}\phi$$

$\hat{=}$ Dirac monopole at $\boldsymbol{k} = 0$, strength eg given by S_z

- return to helicity basis

$$\Rightarrow \boldsymbol{A}_A = R^\dagger(\boldsymbol{k}) \boldsymbol{A}'_A R(\boldsymbol{k}) + i \; \partial_{\boldsymbol{k}} R^\dagger(\boldsymbol{k}) \; R(\boldsymbol{k})$$

Topological invariants

Non-trivial gauge connection $\boldsymbol{A}_A$ in helicity basis

$\Rightarrow$ non-vanishing curvature $\hat{=}$ field strength (monopole $eg = \lambda$)

$$F_{ij} \equiv i[D_i, D_j] = \varepsilon_{ijk} \lambda \frac{k_k}{k^3}$$

- reminder: $H^{\text{eff}} = \frac{\hbar^2 k^2}{2m_\lambda} + V(\boldsymbol{x})$ and $x_i = D_i = i\partial/\partial k_i - A_i(\boldsymbol{k})$
- non-trivial commutation relations

$$[k_i, k_j] = 0, \quad [x_i, k_j] = i\ \delta_{ij}, \quad [x_i, x_j] = -i\ F_{ij}$$

$\Rightarrow$ equations of motion

$$\hbar\dot{k}_i = eE_i \quad \text{and} \quad \dot{x}_i = \frac{\hbar k_i}{m_\lambda} + F_{ij}\dot{k}_j$$

Topological term in equation of motion

- $\dot{x}_i = \frac{\hbar k_i}{m_\lambda} + \frac{e}{\hbar} F_{ij} E_j$ topological term
- $F_{ij} = i\left[D_i, D_j\right] = \epsilon_{ijk} \lambda \frac{k_k}{k^3}$

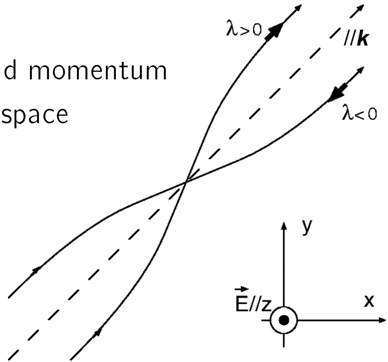
  $\Rightarrow$ noncollinearity of velocity and momentum

  $\hat{=}$ Lorentz force in momentum space

- real-space trajectory:

  shift perpendicular to ***S***

  $\Rightarrow$ spin current
  perpendicular to ***S*** and ***E***



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Spin current in AA

Define: electric field in z-direction, spin parallel to x-axis

$\Rightarrow$ spin current in y-direction j_y^x

- for heavy holes:

$$j_y^{xH} = \frac{\hbar}{3} \sum_{\lambda=\pm 3/2, \boldsymbol{k}} \dot{y} S_x n^\lambda(\boldsymbol{k}) = \frac{\hbar}{3} \sum_{\lambda=\pm 3/2, \boldsymbol{k}} \dot{y} \frac{\lambda k_x}{k} n^\lambda(\boldsymbol{k})$$

$n^\lambda(\boldsymbol{k})$: filling of holes in band λ

use $\dot{y} = F_{yj} \dot{k}_j = \frac{\lambda k_x}{k^3} \frac{eE_z}{\hbar}$ and $S_x = \frac{\lambda k}{k_x}$

- analogous for light holes ($\lambda = \pm 3/2 \;\rightarrow\; \lambda = \pm 1/2$)

$$\Rightarrow \; j_y^x = j_y^{xH} + j_y^{xL} = \frac{eE_z}{36\pi^2}(9k_F^{\mathrm{H}} + k_F^{\mathrm{L}}) \qquad (T = 0)$$

Non-Abelian corrections

Like in AA:

- heavy holes: $j_y^{xH} = \frac{\hbar}{3} \sum_{\lambda=\pm 3/2, \boldsymbol{k}} \dot{y} \frac{\lambda k_x}{k} n^\lambda(\boldsymbol{k})$
- $\dot{y} = F_{yj} \dot{k}_j$

Effect of non-Abelian gauge connection:

$$F_{ij} = i \left[D_i, D_j\right] = \frac{\partial}{\partial k_i} A_j - \frac{\partial}{\partial k_j} A_i + i \left[A_i, A_j\right]$$

$$\Rightarrow F_{ij} = \epsilon_{ijk} \lambda \left(2\lambda^2 - \frac{7}{2}\right) \frac{k_k}{k^3}$$

correction factor of -3 only in the LH band

$\Rightarrow$ spin current

$$j_y^x = \frac{eE_z}{12\pi^2} (3k_F^{\mathrm{H}} - k_F^{\mathrm{L}})$$

Spin conductivity

Quanten Hall Effect:

- Generalization to 4 dimensions:
  electric field E_ν induces $SU(2)$ spin current

$$j^i_\mu = \sigma \eta^i_{\mu\nu} E_\nu \qquad (\mu, \nu = 1, 2, 3, 4, \;\; i = 1, 2, 3)$$

$\eta^i_{\mu\nu}$: t'Hooft tensor

σ: dissipationless transport coefficient

- restriction of E_ν and j^i_μ to 3-dim. subspace:

$\Rightarrow$ dissipationless response

$$j^i_j = \sigma_S \epsilon^{ijk} E_k$$

σ_S: spin conductivity

Spin conductivity: p-GaAs

$$j_j^i = \sigma_S \epsilon^{ijk} E_k$$

spin current including non-Abelian corrections

$$j_y^x = \frac{eE_z}{12\pi^2}(3k_F^{\mathrm{H}} - k_F^{\mathrm{L}}) = \frac{\hbar}{2e}\sigma_S E_z$$

$\Rightarrow$ expression for spin conductivity

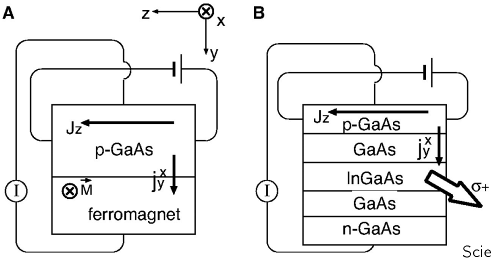
- determined purely by gauge curvature in momentum space
- independent of mean free path or relaxation rates
- finite temperature:
  modification only by Fermi distribution function $n^{\lambda}(\boldsymbol{k})$
- $\Delta E_{\mathrm{HH-LH}} \approx 0.1$ eV $\gg 0.025$ eV

  $\Rightarrow$ effect of same order at room temperature
- for $n = 10^{19}\mathrm{cm}^{-3}$: $\sigma_S \approx \sigma_O$, for $n = 10^{16}\mathrm{cm}^{-3}$: $\sigma_S > \sigma_O$

Detection of spin current

$E, \ J_z \parallel \hat{z}$

$\Rightarrow j_y^x$



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A) Electric transport

- ferromagnet attached to $+y$
- magnetization $\boldsymbol{M}$ along $\pm x$
- lead between $+y$ and $-y$
- measure $I(+x)/I(-)$

B) Polarization of light

- quantum well structure
- sandwiches by p-and n-GaAs
- recombination with e^-
- emission of σ^+ polarized light

Literature

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