Two Dimensional Chern Insulators, the Qi-Wu-Zhang and Haldane Models

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Contents

Dimensional Extension and Reduction

- Foundation of QWZ model is the adiabatic smooth pump sequence of the Rice-Mele model

- Transformed from a 1D time dependant Hamiltonian to a 2D in bulk momentum space

  - Cyclic parameter in a continuous ensemble is Promoted to a momentum parameter

$$\hat{H}(k,t) \rightarrow \hat{H}(k_x,k_y)$$

QWZ Model

$$\hat{H}(k_x, k_y) = \sin k_x \hat{\sigma}_x + \sin k_y \hat{\sigma}_y + [u + \cos k_x + \cos k_y]\hat{\sigma}_z$$

$$\mathbf{d}(k_x, k_y) = \begin{pmatrix} \sin k_x \\ \sin k_y \\ u + \cos k_x + \cos k_y \end{pmatrix}$$

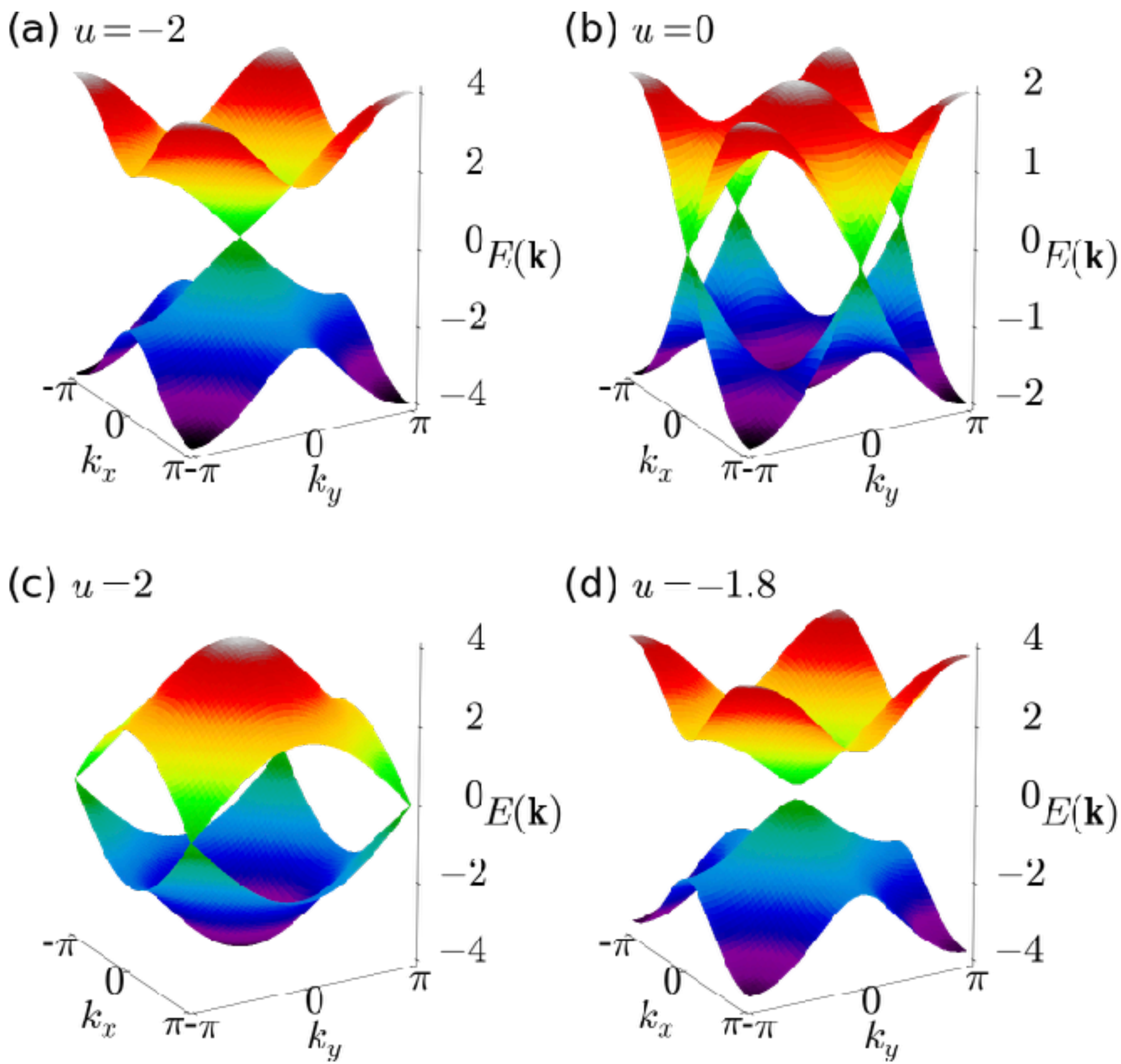
- Gained from the transformations of the adiabatic pumping RM model:
    - Dimensional promotion of time
    - Average intercell hopping to a staggered onsite potential
    - Unitary rotation of the internal Hilbert space

Bulk Dispersion Relation

- Spectrum has two bands and may be determined as follows from the algebraic properties of the Pauli matrices

$$E_{\pm}(k_x, k_y) = \pm\sqrt{\sin^2 k_x + \sin^2 k_y + (u + \cos k_x + \cos k_y)^2}$$

- Tuneable energy gap in spectrum
  - Bands touch at *Dirac Points* when $u = 0, \pm 2$
  - Spectrum gapped at all other values, giving the topological properties we will be discussing



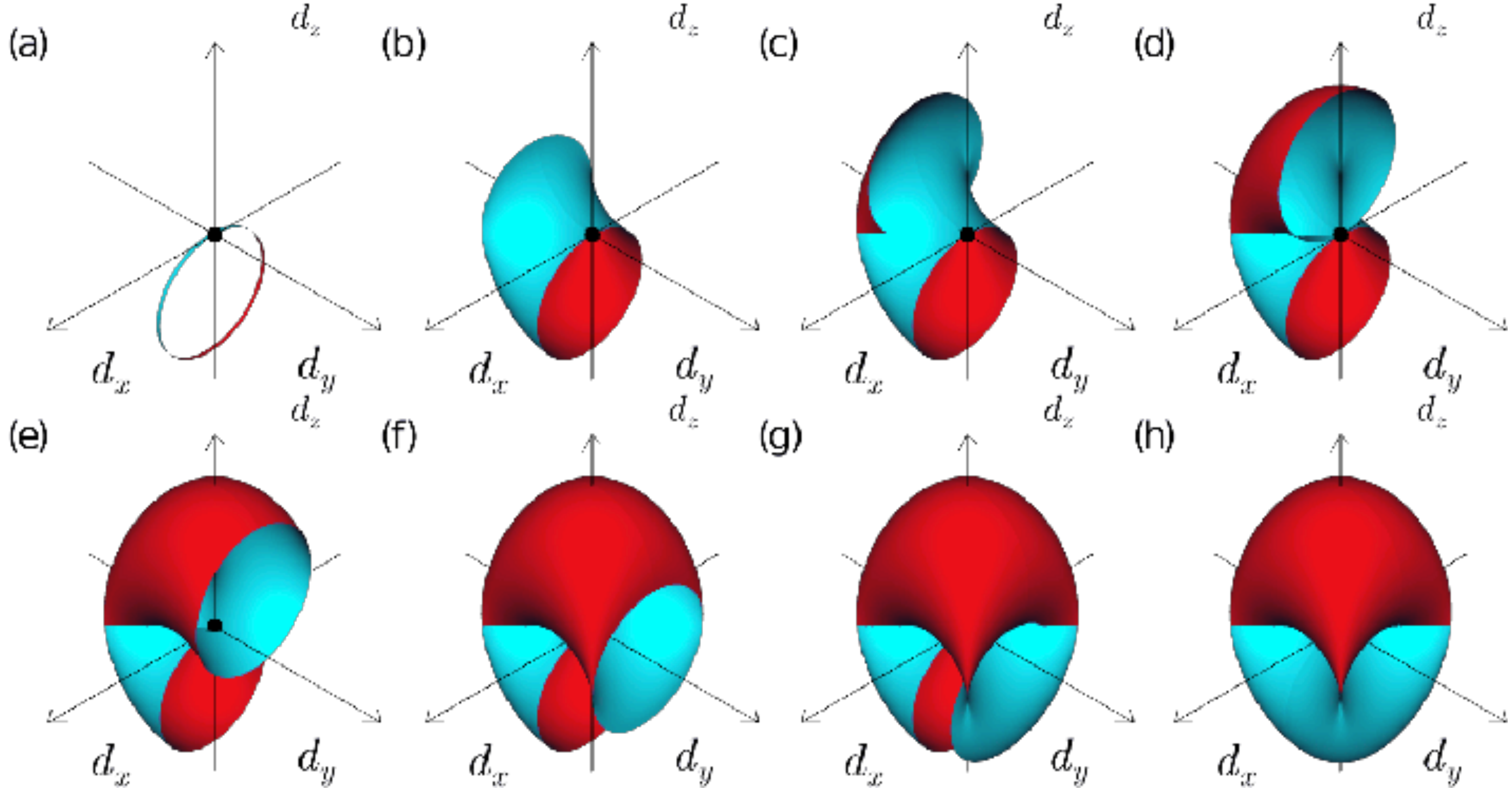
(a) u = −2
(b) u = 0
(c) u −2
(d) u −−1.8
E(k)
4
2
0
−2
−4
1
−1
-π
0
π
k_x
k_y

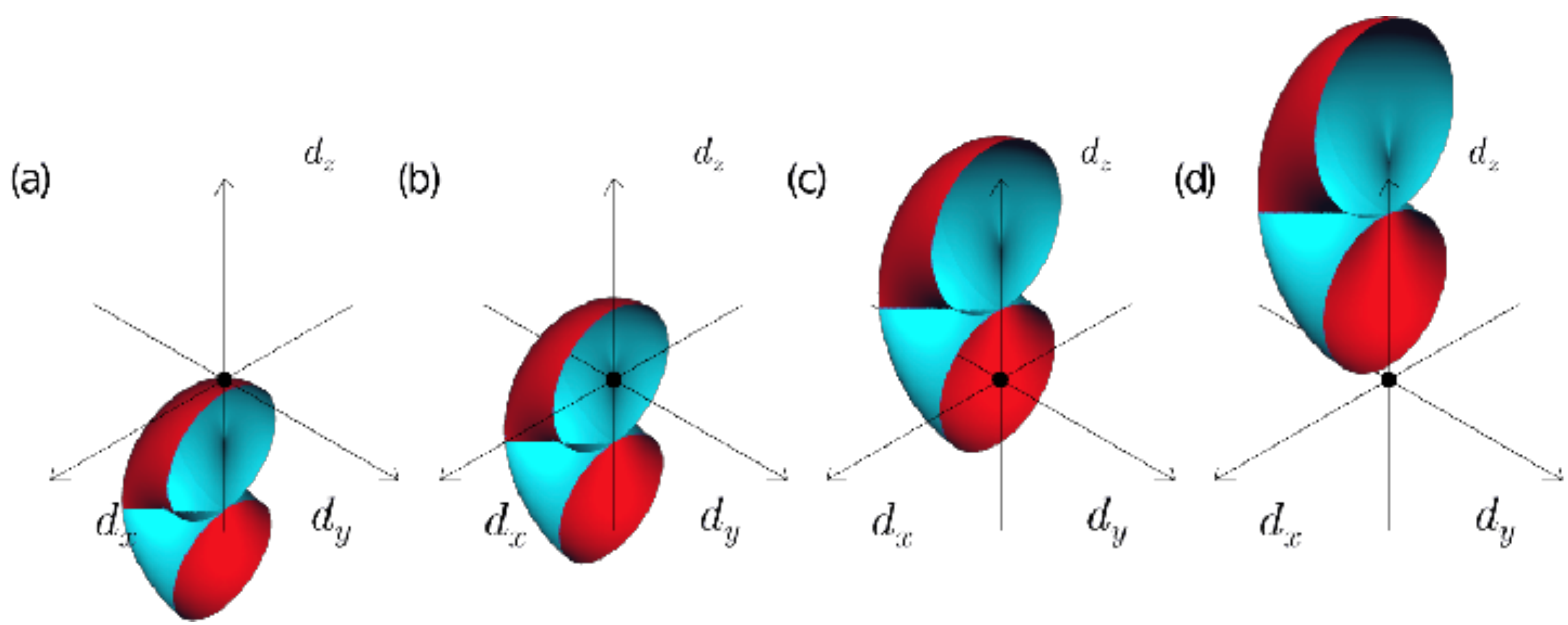
Chern Number of the QWZ Model

- May be counted graphically as the number of times the torus of the image of the BZ in contains the origin

  - This is dependant on the onsite potential parameter such that

$$\begin{aligned} u > 2 \quad &: \quad Q = 0 \\ 0 < u < 2 \quad &: \quad Q = +1 \\ -2 < u < 0 \quad &: \quad Q = -1 \\ u < -2 \quad &: \quad Q = 0 \end{aligned}$$


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
d_z
d_x
d_y


(a)
(b)
(c)
(d)
d_z
d_x
d_y

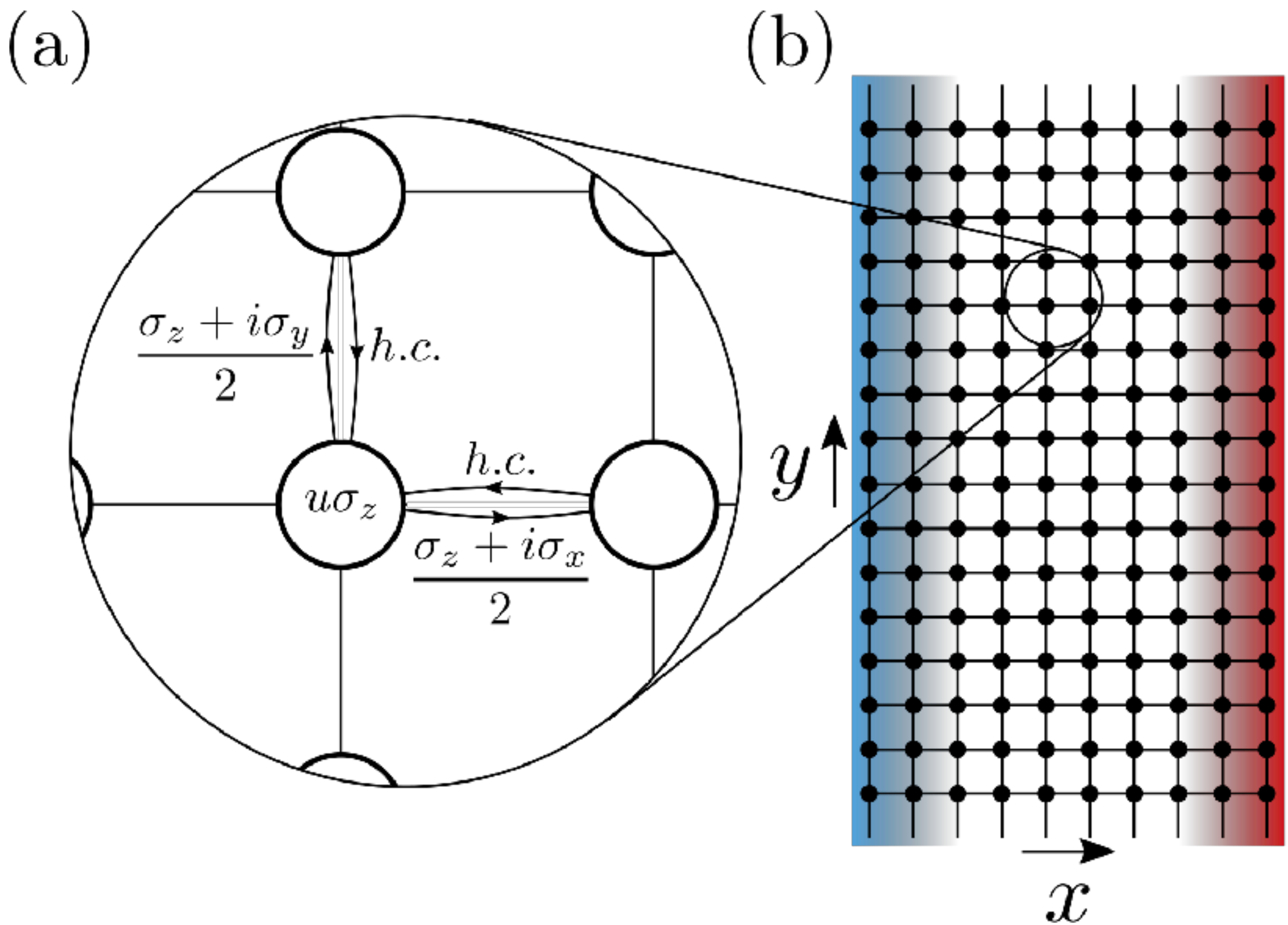
Spatial Hamiltonian

- Inverse Fourier transfer of the bulk momentum space hamiltonian gives the real-spatial hamiltonian

$$\hat{H} = \sum_{m_x=1}^{N_x-1} \sum_{m_y=1}^{N_y} \left(|m_x+1, m_y\rangle \langle m_x, m_y| \otimes \frac{\hat{\sigma}_z + i\hat{\sigma}_x}{2} + h.c. \right)$$

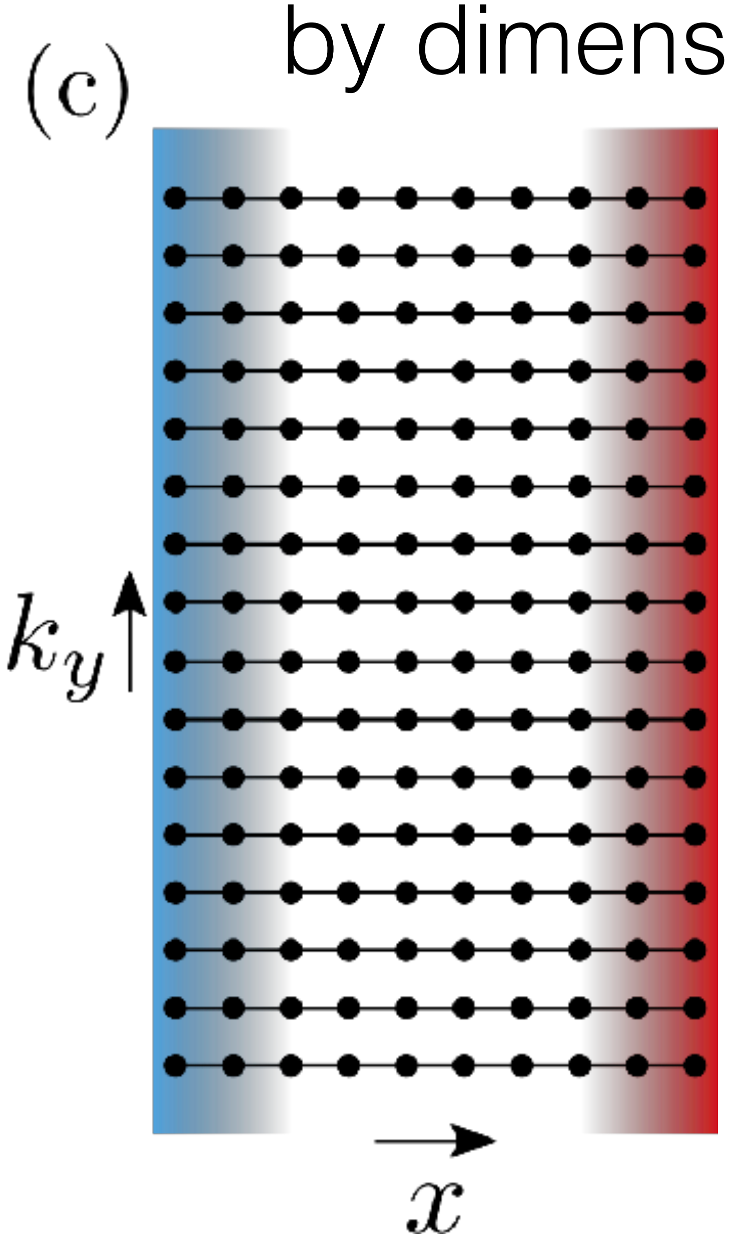
$$+ \sum_{m_x=1}^{N_x} \sum_{m_y=1}^{N_y-1} \left(|m_x, m_y+1\rangle \langle m_x, m_y| \otimes \frac{\hat{\sigma}_z + i\hat{\sigma}_y}{2} + h.c. \right)$$

$$+u \sum_{m_x=1}^{N_x} \sum_{m_y=1}^{N_y} |m_x, m_y\rangle \langle m_x, m_y| \otimes \hat{\sigma}_z$$


(a)
(b)
$\frac{\sigma_z + i\sigma_y}{2}$
$h.c.$
$u\sigma_z$
$h.c.$
$\frac{\sigma_z + i\sigma_x}{2}$
y
x

Edge States

- Since the QWZ model is a transformed adiabatic charge pump, edge states are present and seen by dimensional reduction



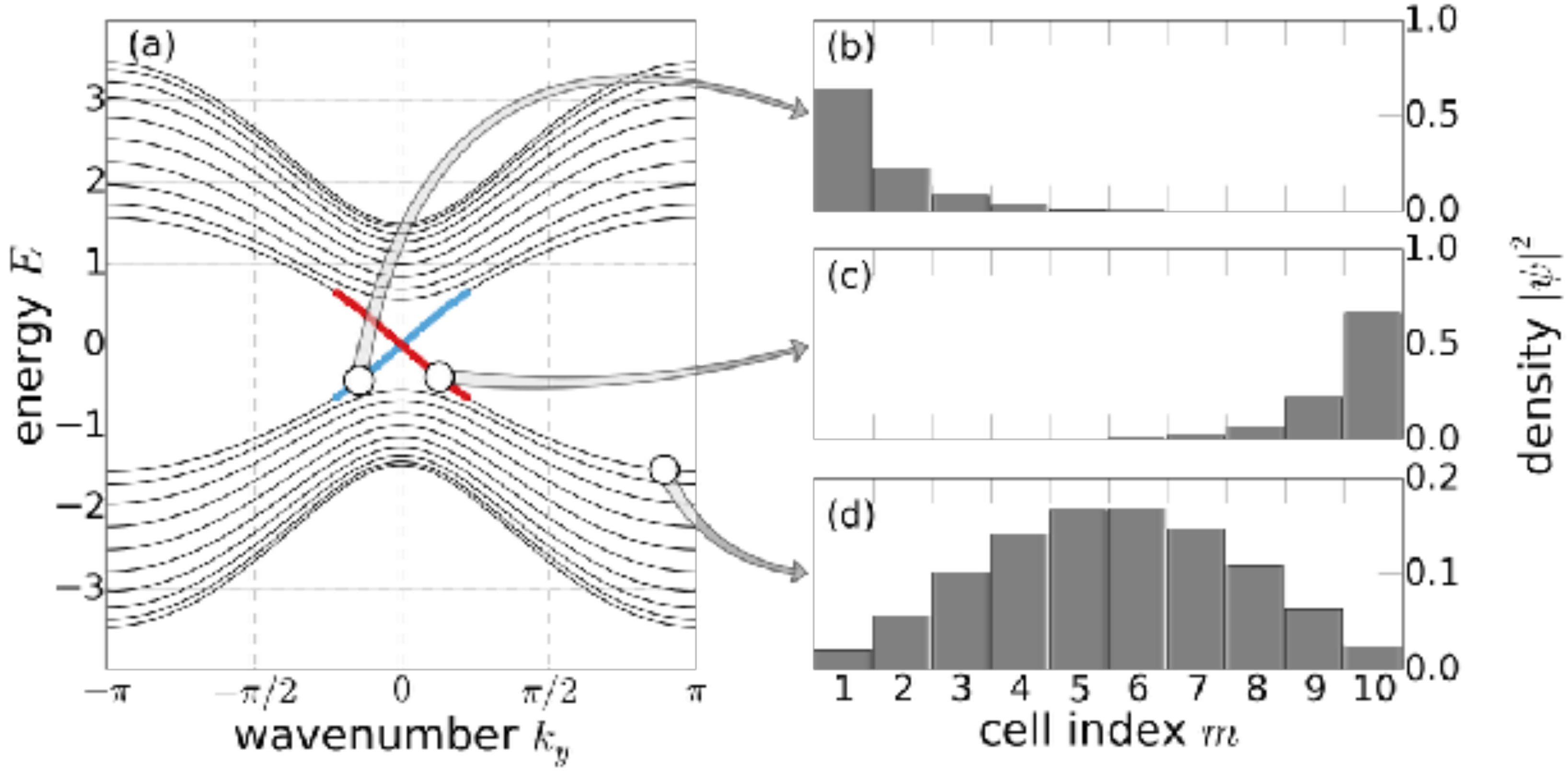
$$\hat{H}(k_y) = \sum_{m_x=1}^{N_x-1} \left(|m_x + 1\rangle \langle m_x| \otimes \frac{\hat{\sigma}_z + i\hat{\sigma}_x}{2} + h.c. \right)$$

$$+ \sum_{m_x=1}^{N_x} |m_x\rangle \langle m_x| \otimes (\cos k_y \hat{\sigma}_z + \sin k_y \hat{\sigma}_y u \otimes \hat{\sigma}_z)$$

Edge States

- In this notation bulk and edge states are present
  - All states are delocalised in y but bulk states are also delocalised in x
  - Similar definitions of edge states in etc RM model may be used in the QWZ model
- Left Edge: $\sum_{m_x=1}^{2} \sum_{\alpha\in\{A,B\}} |\langle\Psi(k_y)|m_x,\alpha\rangle|^2 > 0.6$
- Right Edge: $\sum_{m_x=N-1}^{N} \sum_{\alpha\in\{A,B\}} |\langle\Psi(k_y)|m_x,\alpha\rangle|^2 > 0.6$

Edge States



- Edge States connect lower and upper band across the band gap

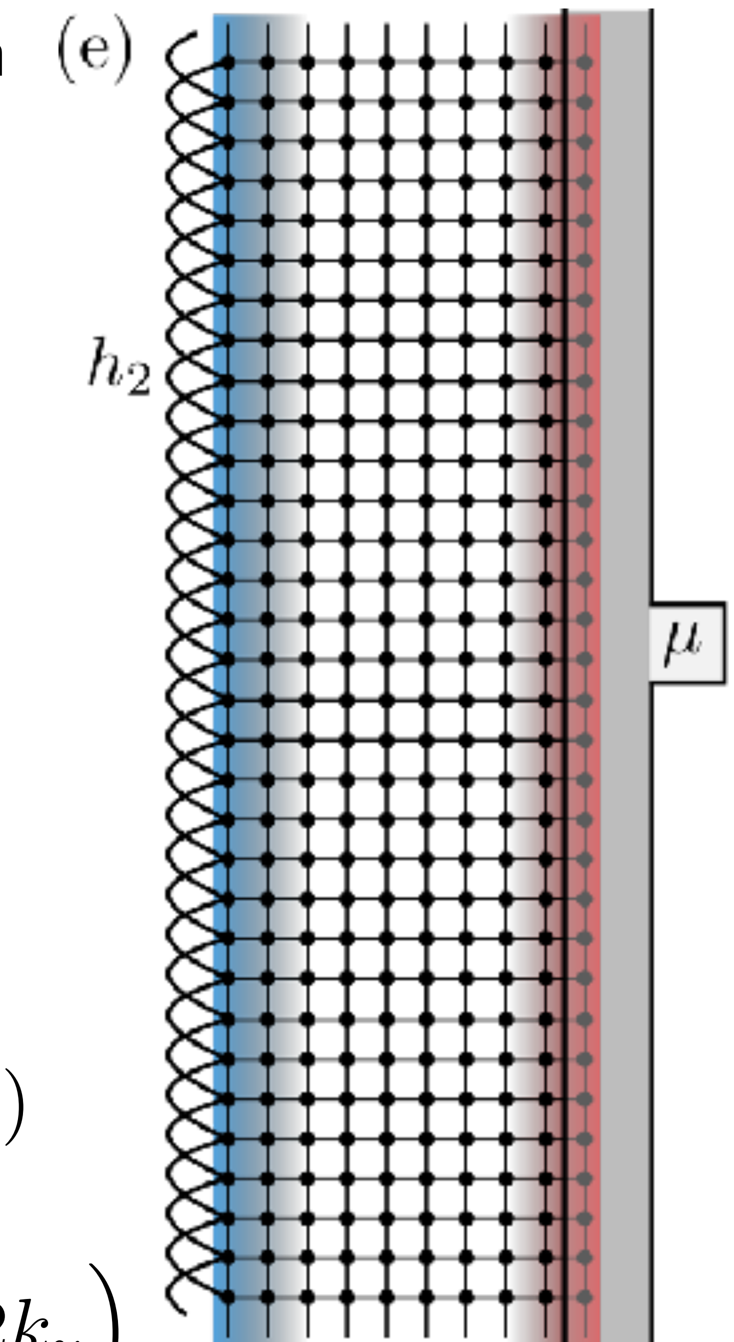
$$Q = N_+ - N_-$$

- Corresponds to unidirectional particle *group velocity* along the edge

- Implicitly, time reversal symmetry is broken when the Chern number is non zero

Edge State Perturbation

- Robustness argument treats the modified edge Hamiltonian with translational invariance along the edges
  - Introduce state independent next nearest neighbour hopping and onsite edge potentials
  - Onsite potential shifts states on the right edge and hopping introduces warping on the left



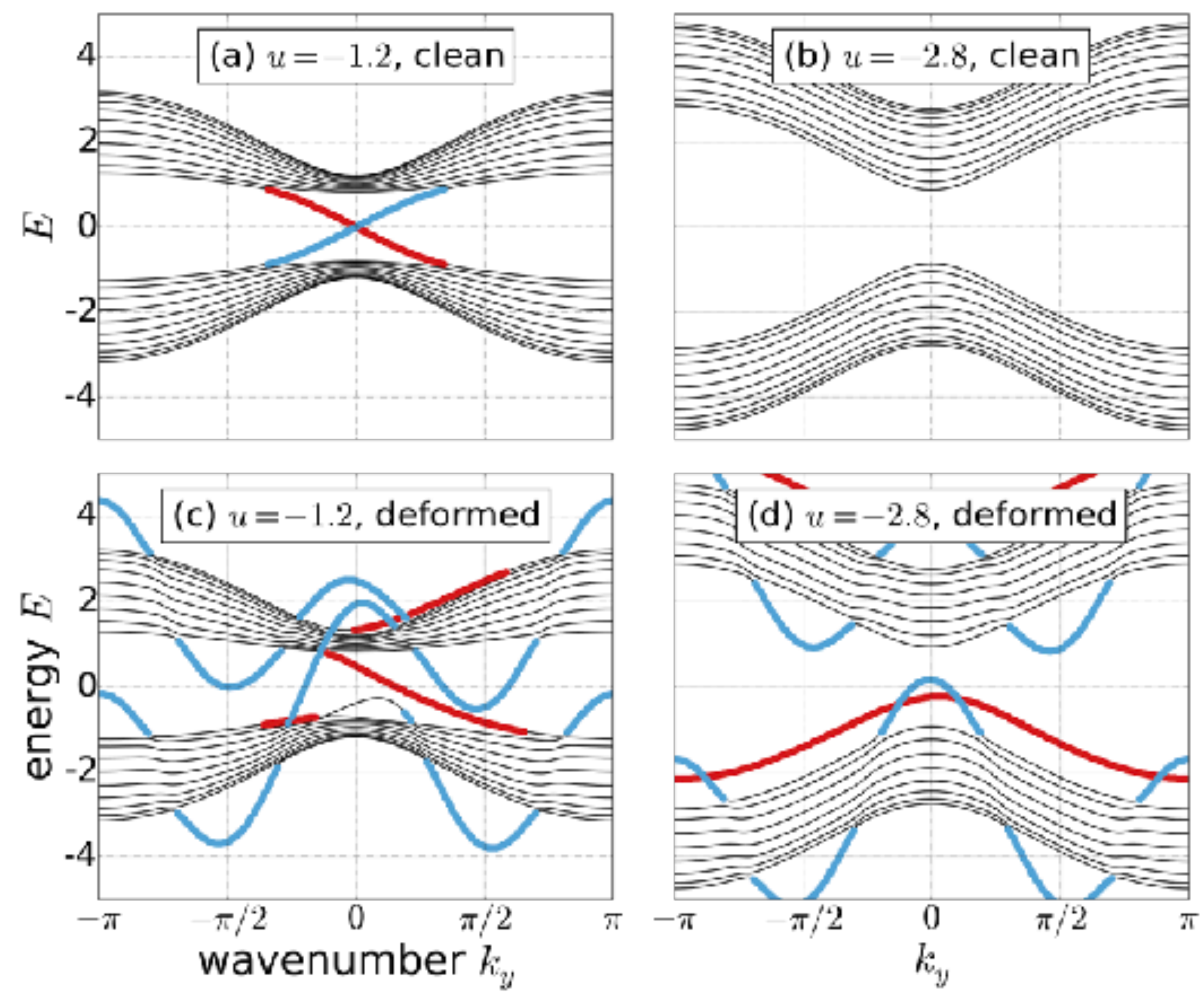
$$\hat{H}(k_y) = \sum_{m_x=1}^{N_x-1} \left(|m_x+1\rangle \langle m_x| \otimes \frac{\hat{\sigma}_z + i\hat{\sigma}_x}{2} + h.c. \right)$$

$$\sum_{m_x=1}^{N_x} |m_x\rangle \langle m_x| \otimes (\cos k_y \hat{\sigma}_z + \sin k_y \hat{\sigma}_y u \otimes \hat{\sigma}_z)$$

$$\sum_{m_x \in \{1,N\}} |m_x\rangle \langle m_x| \otimes \hat{\mathbb{I}}_2 \left(\mu^{(m_x)} + h_2^{(m_x)} \cos 2k_y \right)$$

Edge State Perturbation

- Edge states move into the gap when perturbations are added
- Deformations may add edge states, but only in pairs
- The Chern number remains a topological invariant

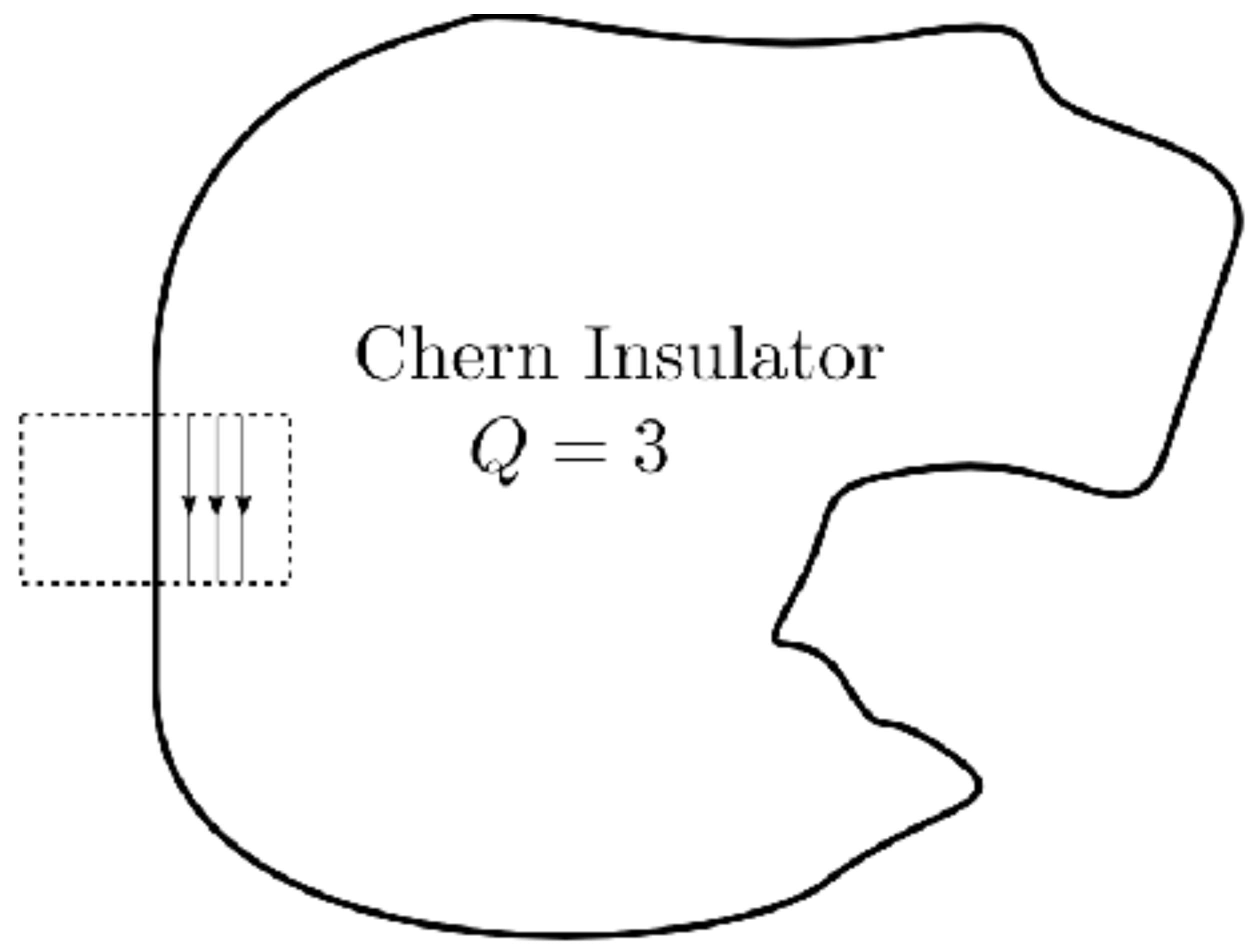


Edge State Robustness

- Before only translationally invariant edges considered

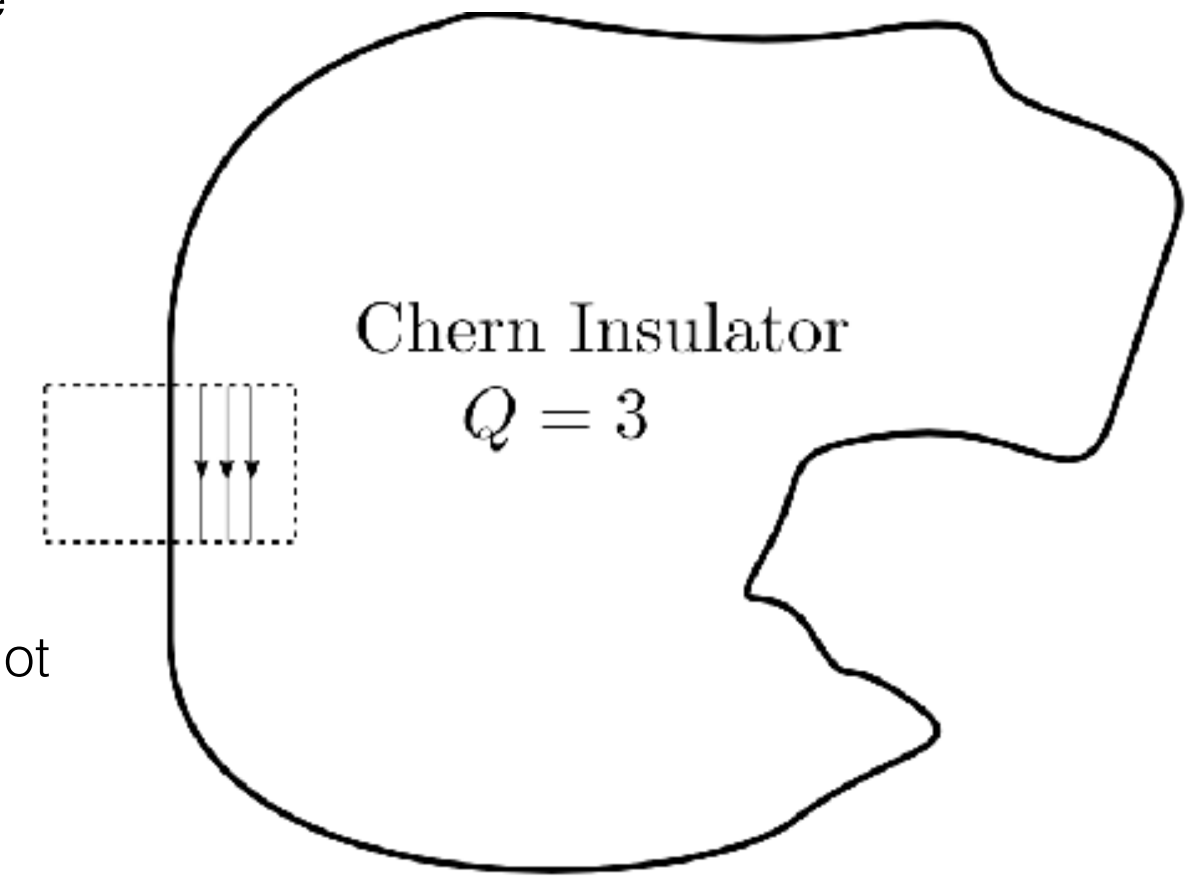
- Without it k_y is no longer a good quantum number

- Edge states must still be present, edge disorder will not close bulk gap only decrease



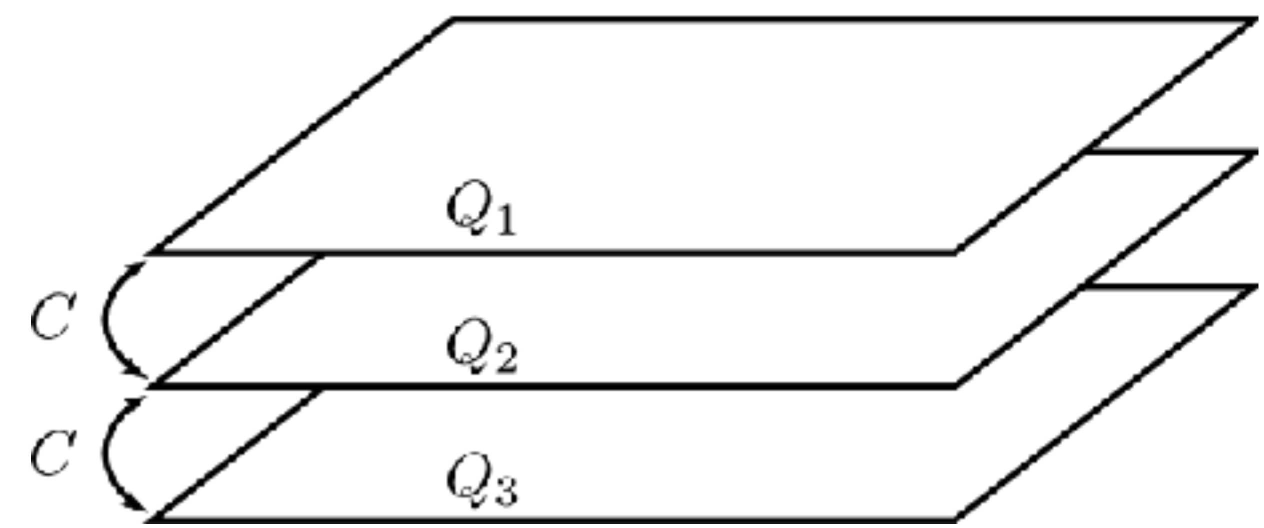
Edge State Robustness

- Smoothy remove disorder
  - Consider a small sample containing some edge
  - Big enough to be considered translationally invariant “Bulk”
  - Here the Hamiltonian can be adiabatically smoothed to an ordered edge
- Unitary group velocity
  - Edge modes are chiral and unitary and so cannot “stop”
  - Particle must complete the loop
- Describes a zero field quantum Hall effect like behaviour of the edge states referred to as the Anomalous Quantum Hall effect (AQHE)

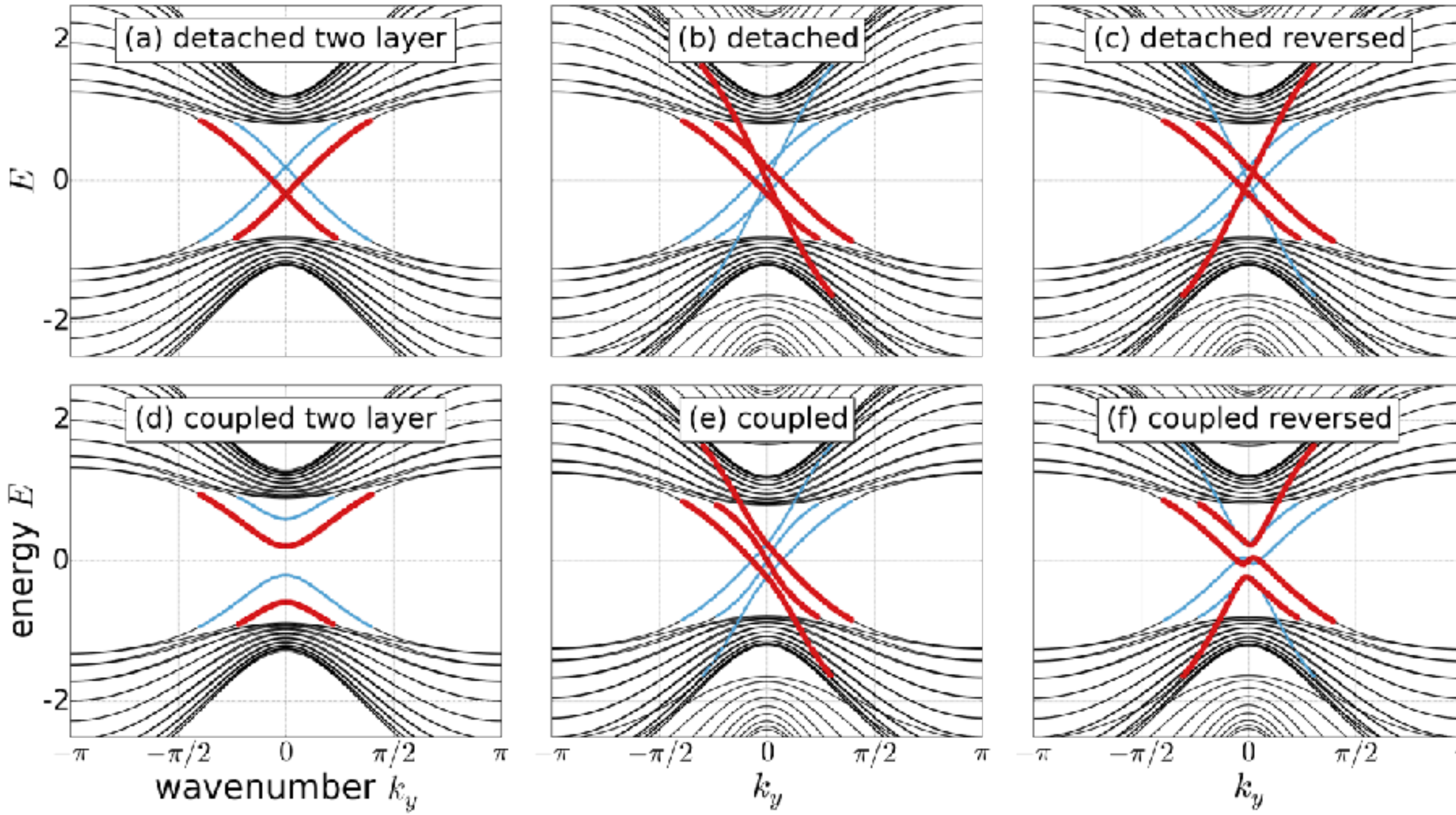


Models with Higher Chern Numbers

- Layer sheets of 2D Chern insulators, single particle Hilbert space then becomes the sum of the layer Hilbert spaces
- Given some state independent coupling between the layers the total Hamiltonian may be deduced
- Coupling of co-propagating edge states lifts degeneracy but only opens a gap when strongly coupled
- Coupling of counter-propagating edge states opens a gap



$$H_3 = \begin{pmatrix} H_{L1} & C\mathbb{I} & 0 \\ C\mathbb{I} & H_{L3} & C\mathbb{I} \\ 0 & C\mathbb{I} & H_{L3} \end{pmatrix}$$



(a) detached two layer
(b) detached
(c) detached reversed
(d) coupled two layer
(e) coupled
(f) coupled reversed
E
energy E
2
0
-2
$-\pi$
$-\pi/2$
0
$\pi/2$
π
wavenumber k_y
k_y

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Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

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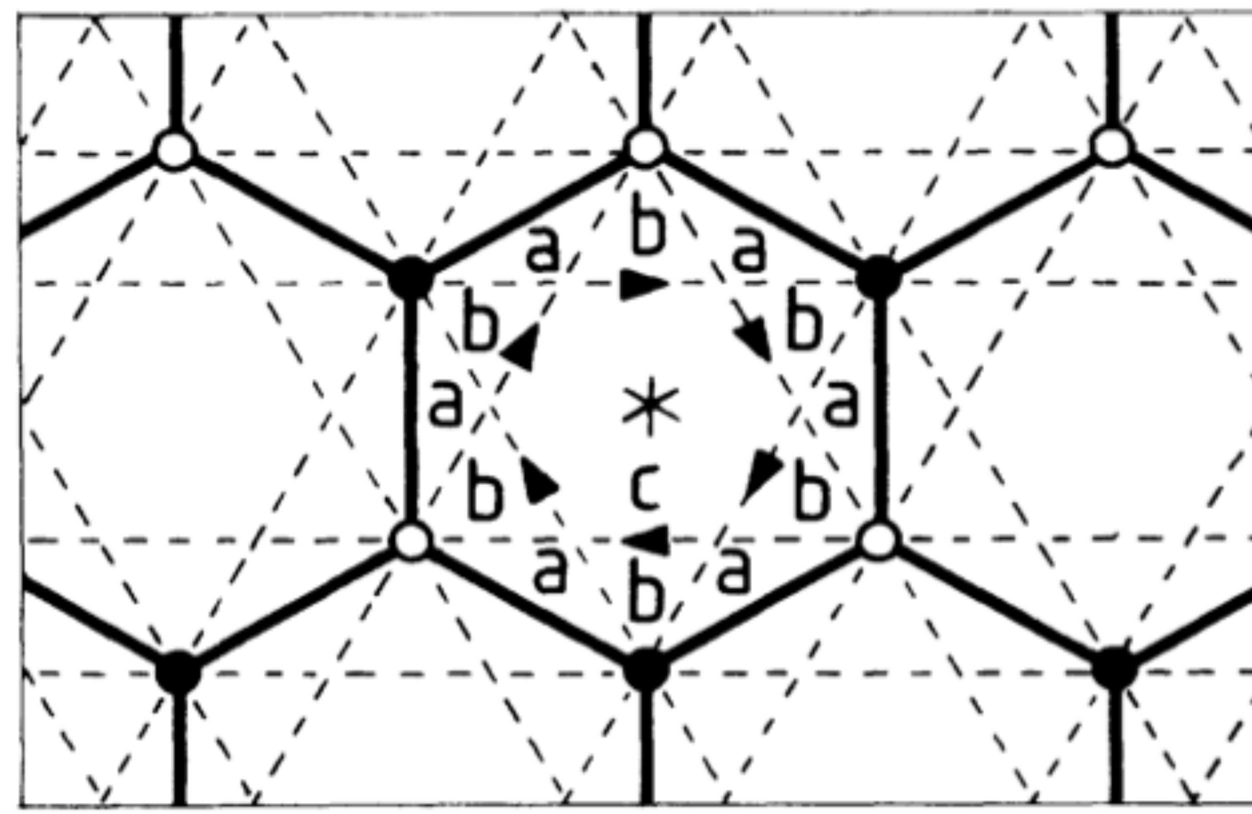
(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.

PACS numbers: 05.30.Fk, 11.30.Rd

Haldane Model Set Up

- Graphene (“2D Graphite”) honeycomb net structure of two interpolating triangular sublattices A and B
- Semi-metal unless there is the addition of an inversion symmetry breaking on site energy term added, then semiconductor
- Next nearest neighbour hopping included
    - Eliminates a particle-hole symmetry
- Period local magnetic flux density added out of plane
    - zero net flux through unit cell
    - Breaks time-reversal symmetry



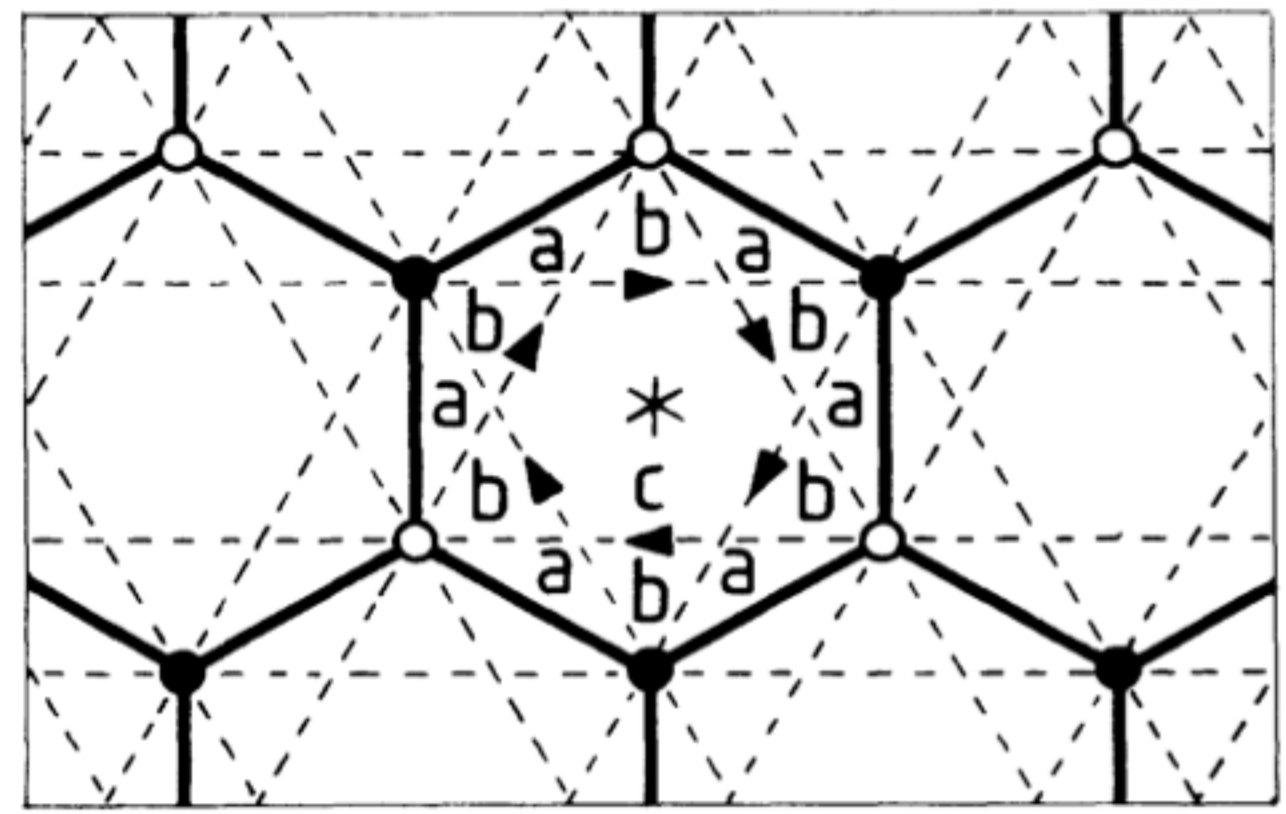
Breaking Time Reversal Symmetry

- No net flux means the vector potential may be chosen to be periodic

- Multiplies the matrix element for hopping between sites by a phase factor

$$exp[i(e/\hbar)\int \mathbf{A}\cdot d\mathbf{r}]$$

- This wont affect the nearest neighbour hopping but the next neighbour terms pick up a phase



$$\phi = 2\pi\frac{2\Phi_a+\Phi_b}{\Phi_0}$$

$$t_2 exp[i\phi]$$

Hamiltonian

$$
\begin{aligned}
H(\mathbf{k}) =& 2t_2 \cos\phi \left(\sum\nolimits_i \cos(\mathbf{k}\cdot\mathbf{b}_i)\right)\mathbb{I} \\
&+ t_1 \left(\sum\nolimits_i [\cos(\mathbf{k}\cdot\mathbf{a}_i)\sigma_x + \sin(\mathbf{k}\cdot\mathbf{a}_i)\sigma_y]\right) \\
&+ \left[M - 2t_2 \sin\phi \left(\sum\nolimits_i \sin(\mathbf{k}\cdot\mathbf{b}_i)\right)\right]\sigma_z
\end{aligned}
$$

- Energy bands only touch at valleys *K (K')* points if:

$$M = \pm 3\sqrt{3}t_2 \sin\phi$$

- Assuming the Fermi level lies between the band gap the hall conductance is quantised at 0 temperature

Energy Solutions

- For some weak external field landau levels are obtained

$$m_\alpha c^2 = M - 3\sqrt{3}\alpha t_2 \sin\phi \qquad\qquad c = \frac{3t_1|\mathbf{a}_i|}{2\hbar}$$

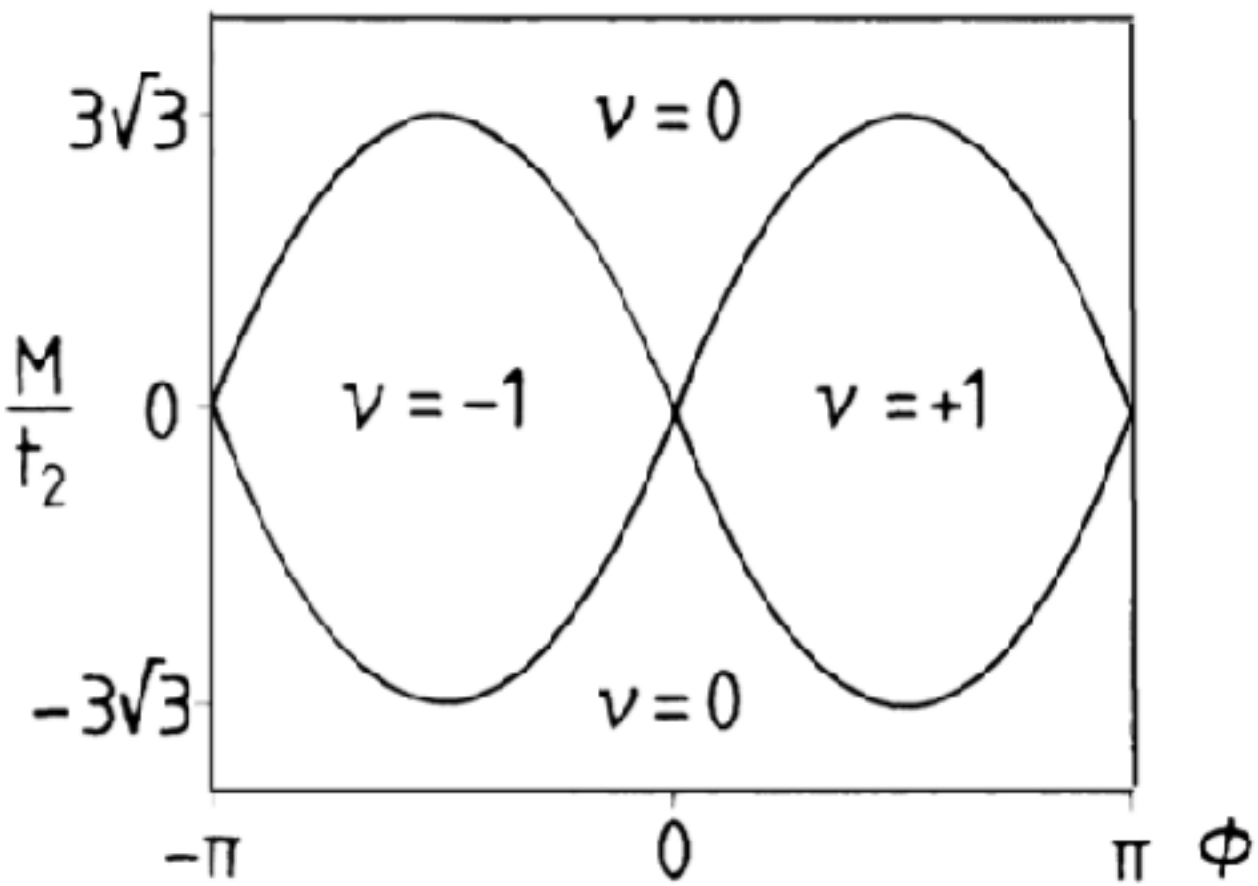
- Zero mode is not symmetric under magnetic flux reversal

$$\epsilon_{\alpha n\pm} = \pm\sqrt{(m_\alpha c^2)^2 + n\hbar|eB_0|c^2} \quad (n \geq 1)$$

$$\epsilon_{\alpha 0} = \alpha m_\alpha c^2 sgn(eB_0)$$

Parity Anomaly Quantum Hall effect

- Term m is set, such that the sign of the parameter is different in different valleys
- Comparison between landau level occupation of this system under external field and a time reversal invariant system shows a difference of 1 level filling
- This system also shows an extra field dependant ground state charge density which may be eliminated at 0 net flux



$$\sigma^{xy} = \frac{\nu e^2}{\hbar}$$

$$\nu = \frac{1}{2}[sgn(m_-) - sgn(m_+)]$$

The Chern Number and Hall Conductivity

- Starting from the *Linear Response Formalism* and using an *Interpolated* Hamiltonian, the Hall conductivity may be calculated

- The Greens functions of the linear response system may be replaced with Projectors given by the interpolation

$$G(i\omega_m, k) = \frac{P_G(k)}{i\omega_m - \epsilon_G} + \frac{P_E(k)}{i\omega_m - \epsilon_E}$$

- Results are only eigenstate and not eigenvalue dependant, indicative of a topological property

The Chern Number and Hall Conductivity

- Hall conductivity shown to be an integral over the filled bands of the Berry Curvature
- Chern number is also defined as the surface integral of the Berry Curvature
- Inversion and time reversal symmetry influence the parity of the Berry Curvature

$$\sigma^{ij} = \int \frac{dk_x dk_y}{(2\pi)^2} \sum_{\alpha=1}^{m} -i(\langle \partial^i(\alpha, k)| \, \partial^j \, |\alpha, k\rangle - \langle \partial^j(\alpha, k)| \, \partial^i \, |\alpha, k\rangle)$$

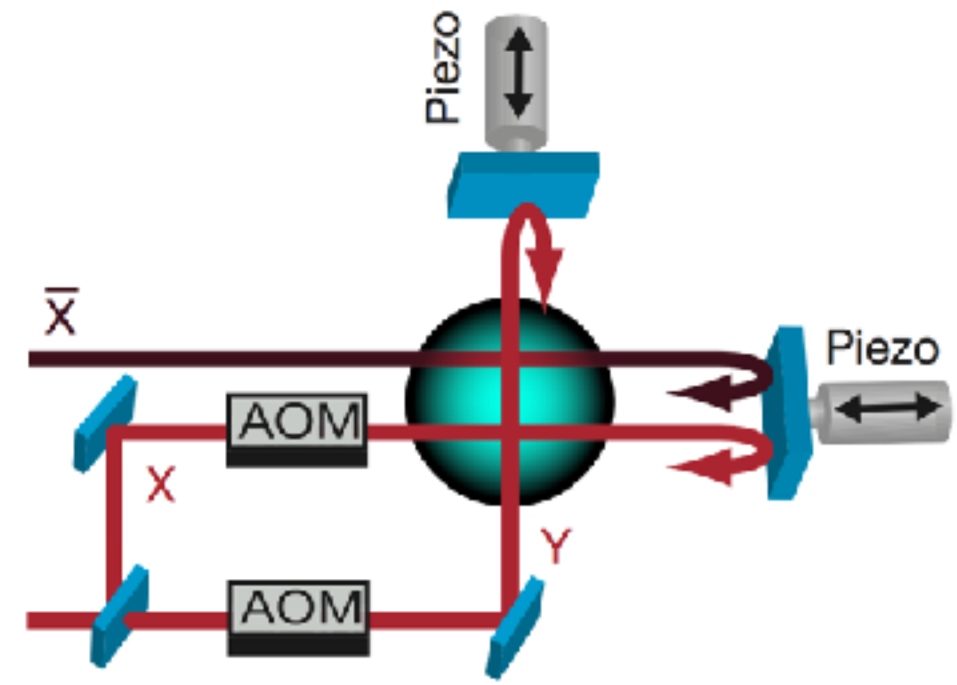
$$\mathbf{\Omega}_n(-\mathbf{k}) = -\mathbf{\Omega}_n(\mathbf{k}) \qquad \mathbf{\Omega}_n(-\mathbf{k}) = \mathbf{\Omega}_n(\mathbf{k})$$

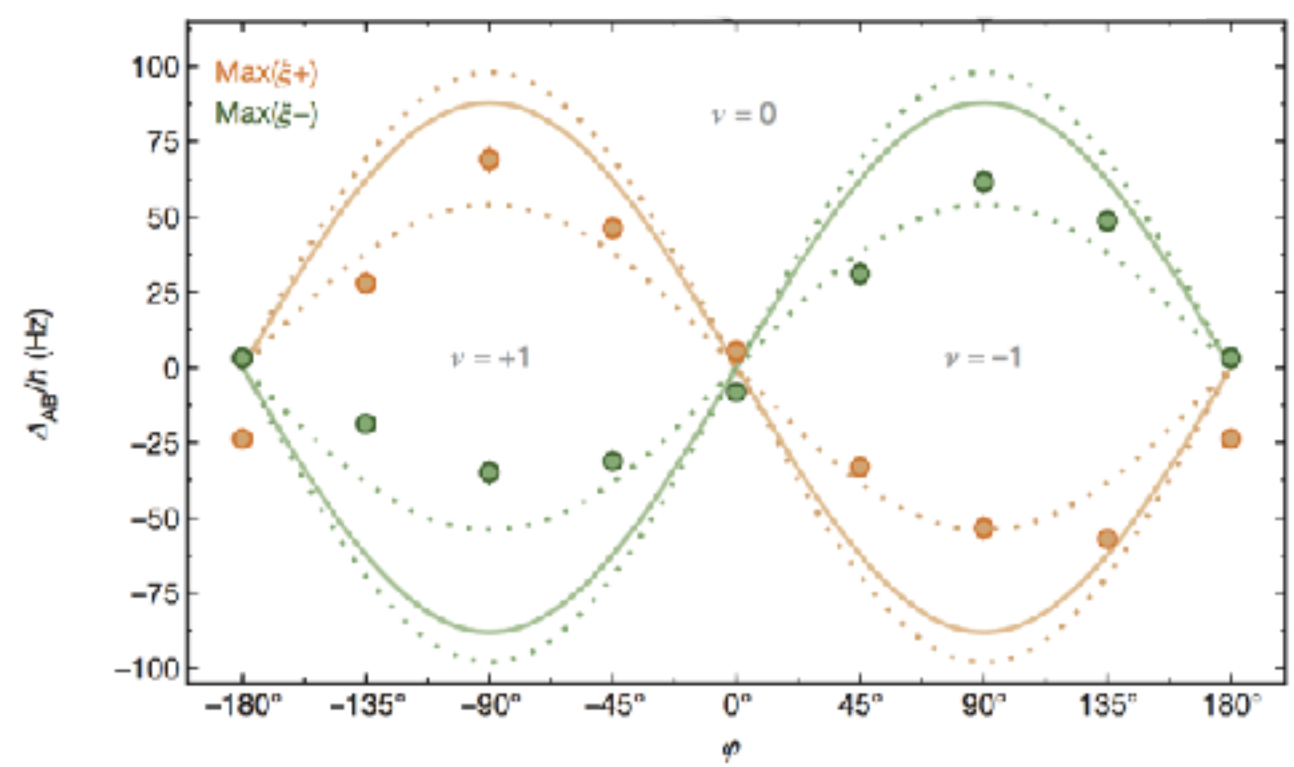
Haldane Experiment

- Jotzu, Gregor, et al. Nature 515.7526 (2014): 237-240

- Ultracold fermionic atoms in a periodically modulated optical honeycomb lattice
- Time reversal symmetry broken by complex next-nearest-neighbour tunnelling induced through circular modulation of the lattice position

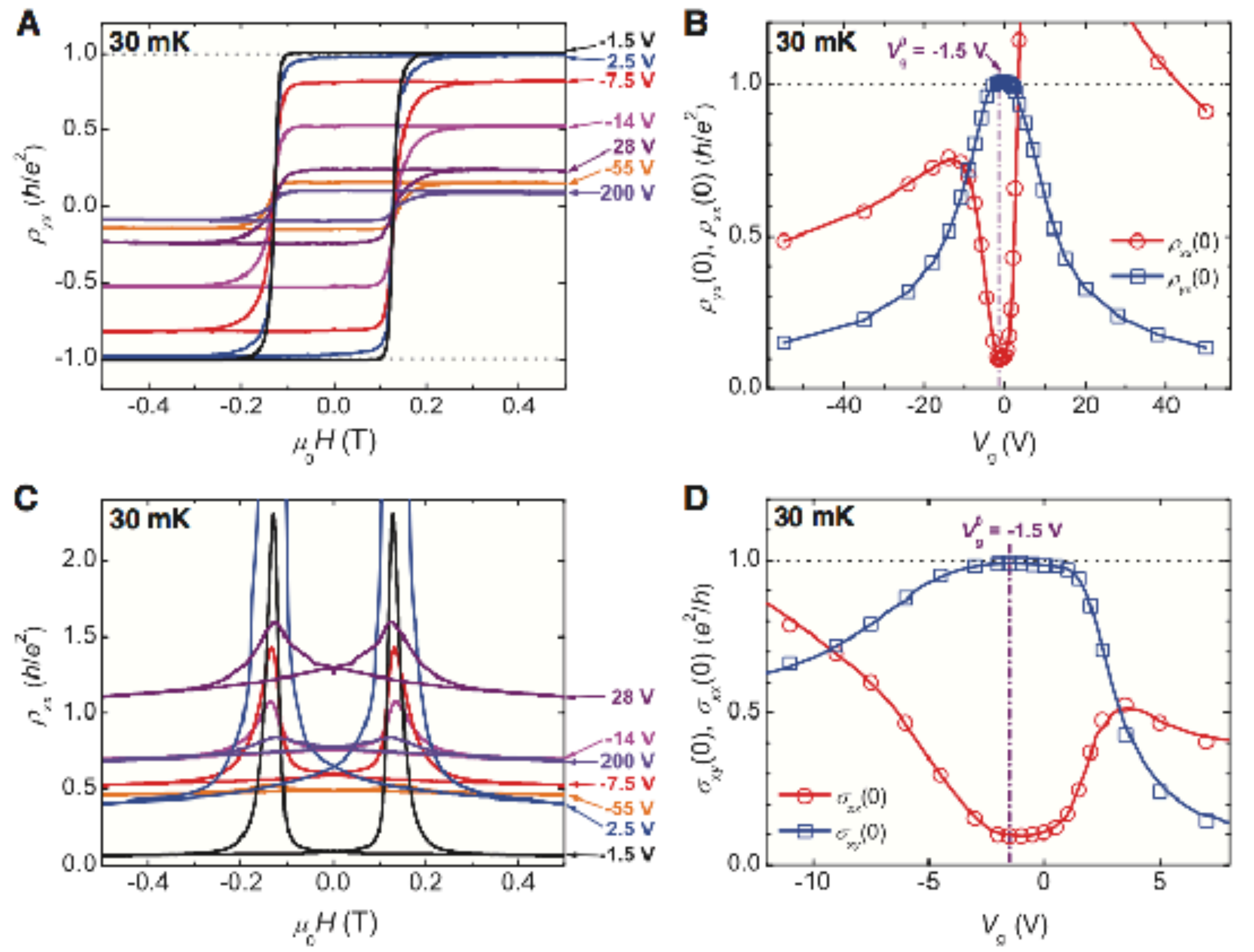
- Inversion symmetry broken by an energy offset between neighbouring sites





AQHE Experimental Realisation

- Chang, Cui-Zu, et al. *Science* 340.6129 (2013): 167-170
- Films of $Cr_{0.15}(Bi_{0.1}Sb_{0.9})_{1.85}Te_3$ on an $SrTiO_3$ dielectric substrate with tuneable chemical potential and ferromagnetic ordering
- Film mobility of these samples are too low for ordinary QHE to be observed



Thank You for Listening, Any Questions?

References

Asbóth, János K., László Oroszlány, and András Pályi. "A Short Course on Topological Insulators." Lecture Notes in Physics, Berlin Springer Verlag. Vol. 919. 2016.

Bernevig, B. Andrei, and Taylor L. Hughes. Topological insulators and topological superconductors. Princeton University Press, 2013.

Chang, Cui-Zu, et al. "Experimental observation of the quantum anomalous Hall effect in a magnetic topological insulator." Science 340.6129 (2013): 167-170.

Haldane, F. Duncan M. "Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the" parity anomaly"." Physical Review Letters 61.18 (1988): 2015.

Jotzu, Gregor, et al. "Experimental realization of the topological Haldane model with ultracold fermions." Nature 515.7526 (2014): 237-240.

Xiao, Di, Ming-Che Chang, and Qian Niu. "Berry phase effects on electronic properties." Reviews of modern physics 82.3 (2010): 1959.