



<http://tinyurl.com/qo2018>

Quantum Optics

Winter semester 2017/2018 - Exercise sheet 15.12.2017

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Problem 1: Phase states of light.

a) Show that the (continuous) phase distribution $P_\alpha(\phi) = |\langle \phi | \alpha \rangle_{\text{coh}}|^2$ for a coherent state $|\alpha\rangle_{\text{coh}}$ is given by:

$$P_\alpha(\phi) = \frac{e^{-|\alpha|^2}}{2\pi} \left| \sum_{n=0}^{\infty} \frac{(e^{-i\phi}\alpha)^n}{\sqrt{n!}} \right|^2.$$

b) Using the following property of the number representation of the phase operator $\hat{\phi} = -i \ln(\widehat{e^{i\theta}})$

$$\langle n | \hat{\phi} | \alpha \rangle = -i \frac{\partial}{\partial n} \langle n | \alpha \rangle$$

and considering $|\alpha| \gg 1$, show that:

$$\langle \alpha | \hat{\phi} | \alpha \rangle = \theta - \frac{i}{2} (\ln \langle n \rangle - \langle \ln(n) \rangle).$$

For $|\alpha| \gg 1$, the Poissonian distribution tends to a vanishing relative uncertainty: $P_\alpha(n) \rightarrow \delta_{n, \langle n \rangle}$. Use this limit to justify that $\langle \hat{\phi} \rangle \approx \theta$. HINT: use the asymptotic expansion of the harmonic number for $n \rightarrow \infty$, $H_n = \sum_{k=1}^n \frac{1}{k} = \ln(n) + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}$, where $\gamma \approx 0.58$ is the Euler-Mascheroni constant and the B_l are the Bernoulli numbers.

c) Using the same assumptions from the previous question, show that:

$$\langle \alpha | \hat{\phi}^2 | \alpha \rangle \approx \theta^2 + \frac{1}{4 \langle n \rangle}.$$

What is the product of the uncertainties $\Delta\phi\Delta n$?

Problem 2: Wigner function.

a) Using the expression for the Wigner function in terms of the density operator, $W(x, p) = \frac{1}{2\pi} \int dx' \langle x - \frac{x'}{2} | \hat{\rho} | x + \frac{x'}{2} \rangle e^{ipx'}$, calculate the Wigner function $W_\alpha(x, p)$ for a coherent state with density matrix $\hat{\rho} = |\alpha\rangle\langle\alpha|$.

b) Calculate the Wigner function $W_\beta(x, p)$ for a thermal state. HINT: use the completeness relations for Hermite polynomials (Mehler's formula).

c) Compare the results of $W_\alpha(x, p)$ and $W_\beta(x, p)$ for $\alpha = 0$ and $\langle n \rangle = 0$. What happens when $\langle n \rangle$ grows?