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Quantum Optics
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Problem 1: Squeezed states of light.

a) Given the two-mode squeezing operator $\hat{S}(\xi_{12}) = \exp(\xi_{12}^* \hat{a}_1 \hat{a}_2 - \xi_{12} \hat{a}_1^\dagger \hat{a}_2^\dagger)$, show that

$$\hat{S}^\dagger(\xi_{12}) \hat{a}_1 \hat{S}(\xi_{12}) = \mu \hat{a}_1 - \nu \hat{a}_2^\dagger \quad \text{and} \quad \hat{S}^\dagger(\xi_{12}) \hat{a}_2 \hat{S}(\xi_{12}) = \mu \hat{a}_2 - \nu \hat{a}_1^\dagger,$$

where $\mu = \cosh(|\xi_{12}|)$, $\nu = e^{i\phi_\nu} \sinh(|\xi_{12}|)$ and $\xi_{12} = e^{i\phi_\nu} |\xi_{12}|$.

b) For a two-mode radiation field strength $\hat{F} = F_1 \hat{a}_1 + \hat{F}_1^* \hat{a}_1^\dagger + F_2 \hat{a}_2 + \hat{F}_2^* \hat{a}_2^\dagger$, where $F_\lambda = |F_\lambda| \exp(i\phi_\lambda)$, show that the (normal ordered) variance of $\hat{F}$ for the state $\hat{S}(\xi_{12})|0\rangle = |\xi_{12}\rangle$ is given by:

$$\langle \xi_{12}| : (\Delta \hat{F})^2 : |\xi_{12}\rangle = 2(|F_1|^2 + |F_2|^2)|\nu|^2 \left[1 - \frac{2|F_1 F_2|}{|F_1|^2 + |F_2|^2} \sqrt{\frac{1 + |\nu|^2}{|\nu|^2}} \cos(\phi_1 + \phi_2 + \phi_\nu) \right].$$

c) Considering $\theta = \phi_1 + \phi_2 + \phi_\nu$ and $|F_1| = |F_2| = 1$, plot this variance as a function of θ for $|\nu| = 1/\sqrt{3}$ and $|\nu| = 1/\sqrt{15}$. How does the squeezing change?

d) Assuming that the squeezing operator can be parametrized in two ways as

$$\hat{S}(\lambda) = \exp\left[\frac{1}{2} \lambda \left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2} \right) \right] = \exp\left[\frac{1}{2} \tau_1(\lambda) \hat{a}^{\dagger 2} \right] \exp\left[\tau_2(\lambda) \left(\hat{n} + \frac{1}{2} \right) \right] \exp\left[\frac{1}{2} \tau_3(\lambda) \hat{a}^2 \right],$$

where $\hat{n} = \hat{a}^\dagger \hat{a}$ and $\lambda \in [0, 1]$, show that

$$\hat{S} \equiv \exp\left[\frac{1}{2} \left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2} \right) \right] = \exp\left[-\frac{\nu}{2\mu} \hat{a}^{\dagger 2} \right] \exp\left[-\ln\mu \left(\hat{n} + \frac{1}{2} \right) \right] \exp\left[\frac{\nu^*}{2\mu} \hat{a}^2 \right],$$

with $\mu = \cosh(|\xi|)$, $\nu = e^{i\phi} \sinh(|\xi|)$ and $\xi = e^{i\phi} |\xi|$.
HINT: use the derivatives of $\hat{S}(\lambda)$ in both forms with respect to λ and compare them (see back page for complementary information).

Problem 2: Schrödinger-Robertson uncertainty relation.

Given an arbitrary state $|\Psi\rangle$ and two arbitrary Hermitian operators $\hat{A}$ and $\hat{B}$, show that, for $\sigma_A = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \Psi \rangle$ and $\sigma_B = \langle \Psi | (\hat{B} - \langle \hat{B} \rangle)^2 | \Psi \rangle$, the following relation is true:

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} \left| \langle \{ \hat{A}, \hat{B} \} \rangle - 2 \langle \hat{A} \rangle \langle \hat{B} \rangle \right|^2 + \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2.$$

Complementary information to problem 1(d). The disentangled form of the squeezing operator allows one to easily see that photons are generated in pairs during the process of parametric downconversion (will be discussed later in the lecture). The operators $\frac{1}{2}\hat{a}^2$ and $\frac{1}{2}\hat{a}^{\dagger 2}$, which appear in the standard form of the single-mode squeezing operator $\hat{S} = \exp\left[\frac{1}{2}\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2}\right)\right]$, and their commutator $\hat{n} + \frac{1}{2}$ form a basis for the Lie algebra of the $SU(1,1)$ group, to which the squeezing operator belongs. Since the elements of a Lie group can be obtained through exponentiation of the elements of its respective algebra, there are two ways of expressing the elements U of a group: $U(\{\alpha\}) = \exp(\sum_i \alpha_i \Gamma_i)$ or $U(\{\beta\}) = \prod_i \exp(\beta_i \Gamma_i)$, where $\alpha_i \neq \beta_i$, $\{\beta\} = \{\beta_1, \ldots, \beta_N\}$ (and equally for $\{\alpha\}$) and Γ_i are the generators (elements of the basis of the algebra). The transformation of the generators $U(\tau_i)\Gamma_j U^{-1}(\tau_i) = \Gamma'_j$ can be found through differentiation of Γ'_j relative to τ_i.