

Quantum non-demolition measurements of a qubit coupled to a harmonic oscillator

Luca Chirolli* and Guido Burkard†

RWTH Aachen University, D-52056 Aachen, Germany

Department of Physics, University of Konstanz, D-78457 Konstanz, Germany

We theoretically describe the weak measurement of a two-level system (qubit) and quantify the degree to which such a qubit measurement has a quantum non-demolition (QND) character. The qubit is coupled to a harmonic oscillator which undergoes a projective measurement. Information on the qubit state is extracted from the oscillator measurement outcomes, and the QND character of the measurement is inferred by the result of subsequent measurements of the oscillator. We use the positive operator value measurement (POVM) formalism to describe the qubit measurement. Two mechanisms lead to deviations from a perfect QND measurement: (i) the quantum fluctuations of the oscillator, and (ii) quantum tunneling between the qubit states $|0\rangle$ and $|1\rangle$ during measurements. Our theory can be applied to QND measurements performed on superconducting qubits coupled to a circuit oscillator.

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I. INTRODUCTION

The possibility to perform repeated quantum measurements on a system with the least possible disturbance was first envisioned in the context of measuring gravitational waves [1]. The application of such a scheme to quantum information has stimulated great interest, in particular in the field of quantum computation, where fast and efficient readout is necessary, and error correction plays an important role [2].

Schemes for QND measurement have been theoretically proposed and experimentally realized in the framework of cavity quantum electrodynamics (cavity-QED), where a superconducting qubit is coupled to a superconducting resonator that behaves as a one mode quantum harmonic oscillator.³⁻⁵ A measurement scheme based on the Josephson bifurcation amplifier (JBA)^{6,7} has been adopted with the aim to perform QND measurements of superconducting qubit^{8,9}. In these experiments a deviation of $\sim 10\%$ from perfect QND behavior has been found.

Motivated by those recent experimental achievements we analyze a measurement technique based on the coupling of the qubit to a driven harmonic oscillator. A quadrature of the harmonic oscillator is addressed via a projective measurement. The qubit that is coupled to the oscillator affects the outcomes of the measurement of the oscillator and information on the qubit state can be extracted from the results of the projective measurement of the oscillator.

We aim to shed some light on the possibilities to perform qubit QND measurements with such a setup, and try to understand whether deviations from the expected behavior could arise from quantum tunneling between the qubit states. Such a tunneling process, although made small compared to the qubit energy splitting, violates the QND conditions.

One of the possible implementations of the system under consideration is the four-junction persistent current qubit⁸ (flux qubit) depicted in Fig 1a). It consists of

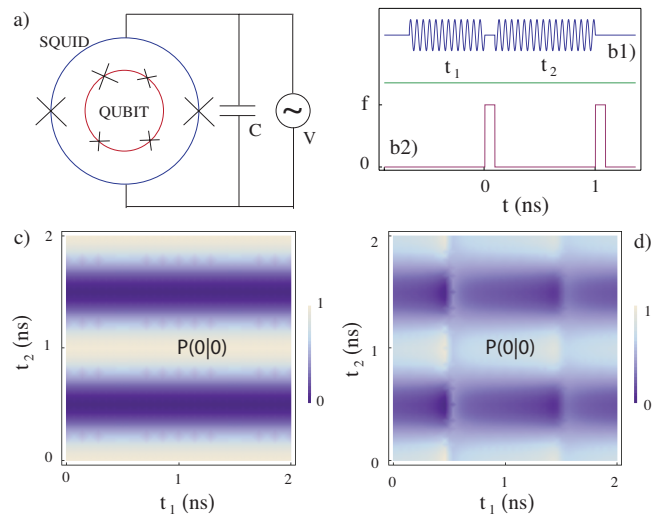


FIG. 1: (Color online) a) Schematics of the 4-Josephson junction superconducting flux qubit surrounded by a SQUID. b) Measurement scheme: b1) two short pulses at frequency $\sqrt{\epsilon^2 + \Delta^2}$, respectively before the first measurement and between the first and the second measurement prepare the qubit in a generic state. Here, ϵ and Δ represent the energy difference and the tunneling amplitude between the two qubit states. b2) Two pulses of amplitude f and duration $t = 0.1$ ns drive the harmonic oscillator to a qubit-dependent state. c) Perfect QND: conditional probability $P(0|0)$ for $\Delta = 0$ to detect the qubit in the state $|0\rangle$ vs t_1 and t_2 , duration of the oscillation at Rabi frequency of 1 GHz. d) Conditional probability $P(0|0)$ for $\Delta t = \Delta/\epsilon = 0.1$. A short qubit relaxation time $T_1 = 10$ ns is assumed.

a superconducting loop with four Josephson junctions and its low temperature dynamics is confined to the two lowest-energy states. For external magnetic flux close to half-integer multiple of $\Phi_0 = h/2e$, the superconducting flux quantum, the two lowest-energy eigenstates are combinations of clockwise and counter clockwise circulating current states. These two states represent the qubit.

The measurement apparatus consists of a superconducting quantum interference device (SQUID), composed by two Josephson junctions, inductively coupled to the qubit loop. The SQUID behaves as a non-linear inductance and together with a shunt capacitance forms a non-linear LC -oscillator, which is externally driven. The two qubit states produce opposite magnetic field that translate into a qubit dependent effective Josephson inductance of the SQUID. The response of the driven SQUID is therefore qubit-dependent.

In order to treat the problem in a full quantum mechanical way, we linearize the SQUID equation of motion, such that the effective coupling between the driven LC -oscillator and the qubit turns out to be quadratic. The qubit Hamiltonian is $\mathcal{H}_S = \epsilon\sigma_Z/2 + \Delta\sigma_X/2$. In the experiment⁸, the tunneling amplitude Δ between the two qubit current states is made small compared to the qubit gap $E = \sqrt{\epsilon^2 + \Delta^2}$, therefore also $\Delta \ll \epsilon$, such that it can be considered as a small perturbation. The absence of the tunneling term would yield a perfect QND Hamiltonian (see below). From the experimental parameters $\Delta = 5$ GHz and $E = 14.2$ GHz^{8,16}, it turns out that $\Delta/\epsilon \approx 0.38$, yielding corrections of the order of 10% at second order.

The QND character of the qubit measurement is studied by repeating the measurement. A perfect QND setup guarantees identical outcomes for the two repeated measurement with certainty. In order to fully characterize the properties of the measurement, we can initialize the qubit in the state $|0\rangle$, then rotate the qubit by applying a pulse of duration t_1 before the first measurement and a second pulse of duration t_2 between the first and the second measurement. The conditional probability to detect the qubit in the states s and s' is expected to be independent of the first pulse, and to show sinusoidal oscillation with amplitude 1 in time. Deviations from this expectation witness a deviation from a perfect QND measurement. The sequence of qubit pulses and oscillator driving is depicted in Fig. 1b). The conditional probability $P(0|0)$ to detect the qubit in the state "0" twice in sequence is plotted versus t_1 and t_2 in Fig. 1c) for $\Delta = 0$, and in Fig. 1d) for $\Delta t = \Delta/\epsilon = 0.1$. We anticipate here that a dependence on Δt_1 is visible when the qubit undergoes a flip in the first rotation. Such a dependence is due to the imperfections of the mapping between the qubit state and the oscillator state, and is present also in the case $\Delta = 0$. The effect of the non-QND term $\Delta\sigma_X$ results in an overall reduction of the $P(0|0)$.

Many attempts to understand the possible origin of the deviations from perfect QND behavior appearing in the experiments have been concerned with the interaction with the environment^{10–16}. However, to our knowledge, the effect of tunneling between the two qubit states, which despite its smallness represents a permanent a non-QND term, has not yet been taken into consideration. The form of the Josephson non-linearity dictates the form of the coupling between the qubit and the linear oscillator, with the qubit coupled to the photon number op-

erator of the driven harmonic oscillator, $\sigma_Z a^\dagger a$, rather than to one quadrature, $\sigma_X(a + a^\dagger)$, and the effect of the tunneling term σ_X present in the qubit Hamiltonian is considered as a small perturbation.

The work we present is not strictly confined to the analysis of superconducting flux qubit measurement. Rather, it is applicable to a generic system of coupled qubit and harmonic oscillator that can find an application in many contexts. Moreover, the analysis we present is based on the general formalism of the positive operator valued measure (POVM), that represents the most general tool in the study of quantum measurements.

The paper is structured as follows: in Sec. II we introduce the idea of QND measurement and describe the conditions under which a QND measurement can be performed. In Sec. III we derive the quadratic coupling between the qubit and the oscillator and the Hamiltonian of the total coupled system. In Sec. IV we construct the qubit single measurement with the POVM formalism and in Sec. V we consider the effect of the non-QND term in the POVM that describes the single measurement. In Sec. VI we construct the two-measurement formalism, by extending the formalism of POVM to the two subsequent measurement case. In Sec. VII we consider the single measurement in the case $\Delta = 0$ and study the condition for having a good QND measurement. In Sec. VIII we calculate the contribution at first order in Δ/ϵ to the POVM and to the outcome probability for the qubit single measurement, and in Sec. IX we calculate the contribution at second order in Δ/ϵ . In Sec. X we calculate the contribution at first and second order in Δ/ϵ to the POVM and to the outcome probability for the qubit two subsequent measurement. In Sec. XI we study the QND character of the measurement by looking at the conditional probability for the outcomes of two subsequent measurements when we rotate the qubit before the first measurement and between the first and the second measurement.

II. QND MEASUREMENTS

We consider a quantum system on which we want to measure a suitable observable \hat{A} . A measurement procedure is based on coupling the system under consideration to a meter. The global evolution entangles the meter and the system, and a measurement of an observable \hat{B} of the meter provides information on the system. In general, a strong projective measurement on the meter translates into a weak non-projective measurement on the system. This is because the eigenstates of the coupled system differ in general from the product of the eigenstates of the measured observable on the system and those of the meter.

Three criteria that a measurement should satisfy in order to be QND have been formulated¹⁰: i) correct correlation between the input state and the measurement result; ii) the action of measuring should not alter the

observable being measured; iii) repeated measurement should give the same result. These three criteria can be cast in a more precise way: *the measured observable \hat{A} must be an integral of motion for the coupled meter and system*¹. Formally this means that the observable \hat{A} that we want to measure must commute with the Hamiltonian \mathcal{H} , that describes the interacting system and meter,

$$[\mathcal{H}, \hat{A}] = 0. \quad (1)$$

Such a requirement represents a sufficient condition in order that an eigenstate of the observable \hat{A} , determined by the measurement, does not change under the global evolution of the coupled system and meter. As a consequence, a subsequent measurement of the same observable \hat{A} provides the same outcome as the previous one with certainty.

Finally, in order to obtain information on the system observable \hat{A} by the measurement of the meter observable \hat{B} , it is necessary that the interaction Hamiltonian does *not* commute with \hat{B} ,

$$[\mathcal{H}_{\text{int}}, \hat{B}] \neq 0, \quad (2)$$

where \mathcal{H}_{int} describes the interaction between the meter and the system,

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_{\text{meter}} + \mathcal{H}_{\text{int}}. \quad (3)$$

Altogether, these criteria provide an immediate way to determine whether a given measurement protocol can give rise to a QND measurement.

III. MODEL: QUADRATIC COUPLING

As far as the application of our model to the measurement of a persistent current qubit with a SQUID is concerned, we provide here a derivation of the quadratic coupling mentioned in the introduction.

The Hamiltonian of a SQUID in an external magnetic field can be written as

$$\mathcal{H} = \frac{\hat{Q}^2}{2C} - \frac{\Phi_0^2}{L_J} \cos(2\pi\Phi/\Phi_0) \cos\hat{\varphi} \quad (4)$$

where $\hat{\varphi} = \hat{\varphi}_1 - \hat{\varphi}_2$ is the difference of the phases of the two Josephson junctions $\hat{\varphi}_1$ and $\hat{\varphi}_2$ that interrupt the SQUID loop, L_J the Josephson inductance of the junctions (nominally equal), and \hat{Q} is the difference of the charges accumulated on the capacitances C that shunt the junctions. Up to a constant factor, $\hat{\varphi}$ and \hat{Q} are canonically conjugate variables that satisfy $[\hat{\varphi}, \hat{Q}] = 2ei$.

The qubit inductively coupled to the SQUID affects the magnetic flux through the loop. Splitting the external flux into a constant term and a qubit dependent term, such that $\cos(2\pi\Phi/\Phi_0) = \cos(2\pi\Phi_{\text{ext}}/\Phi_0 + 2\pi MI_q \sigma_Z/\Phi_0) \equiv \lambda_0 + \lambda_1 \sigma_Z$, with I_q the current in the qubit loop and M the mutual inductance between qubit

and SQUID loop. Expanding the potential up to second order in $\hat{\varphi}$, one obtains

$$\mathcal{H} \approx \frac{\hat{Q}^2}{2C} + (\lambda_0 + \lambda_1 \sigma_Z) \left(\frac{\Phi_0}{2\pi} \right)^2 \frac{\hat{\varphi}^2}{2L_J}. \quad (5)$$

with $\lambda_0 = \cos(2\pi\Phi_{\text{ext}}/\Phi_0) \cos(2\pi MI_q/\Phi_0)$ and $\lambda_1 = -\sin(2\pi\Phi_{\text{ext}}/\Phi_0) \sin(2\pi MI_q/\Phi_0)$. We introduce the zero point fluctuation amplitude $\sigma = (L_J/\lambda_0 C)^{1/4}$, the bare harmonic oscillator frequency $\omega_{\text{ho}} = \sqrt{\lambda_0/L_J C}$, and the in-phase and in-quadrature components of the field

$$\frac{\Phi_0}{2\pi} \hat{\varphi} \equiv \hat{X} = \sigma \sqrt{\frac{\hbar}{2}} (a + a^\dagger), \quad (6)$$

$$\hat{Q} \equiv \hat{P} = -\frac{i}{\sigma} \sqrt{\frac{\hbar}{2}} (a - a^\dagger), \quad (7)$$

with a and a^\dagger harmonic oscillator annihilation and creation operators satisfying $[a, a^\dagger] = 1$. From this follows $[\hat{X}, \hat{P}] = i\hbar$. Apart from a renormalization of the qubit splitting, the Hamiltonian of the coupled qubit and linearized SQUID turns out to be

$$\mathcal{H} = \hbar\omega_{\text{ho}}(1 + \tilde{g}\sigma_Z)a^\dagger a + \hbar\tilde{g}\sigma_Z(a^2 + a^{\dagger 2}), \quad (8)$$

with $\tilde{g} = \lambda_1/2\lambda_0 = \tan(2\pi\Phi_{\text{ext}}/\Phi_0) \tan(2\pi MI_q/\Phi_0)/2$. The frequency of the harmonic oscillator describing the linearized SQUID is then effectively split by the qubit.

The effective quantum Hamiltonian of the system composed of a qubit and a driven harmonic oscillator coupled by the quadratic Hamiltonian Eq. (8) is ($\hbar = 1$)

$$\mathcal{H}(t) = \mathcal{H}_S + \mathcal{H}_{\text{meter}} + \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{drive}}(t). \quad (9)$$

The qubit Hamiltonian written by means of the Pauli matrices σ_i (we denote 2x2 matrices in qubit space with bold symbols) in the basis of the current states $\{|0\rangle, |1\rangle\}$ is

$$\mathcal{H}_S = \frac{\epsilon}{2} \sigma_Z + \frac{\Delta}{2} \sigma_X, \quad (10)$$

where $\epsilon = 2I_q(\Phi_{\text{ext}} - \Phi_0/2)$ represents an energy difference between the qubit states and Δ the tunneling term between these states. The Hamiltonian of the oscillator (or SQUID) is

$$\mathcal{H}_{\text{meter}} = \omega_{\text{ho}} a^\dagger a. \quad (11)$$

The Hamiltonian that describes the coupling between the qubit and the harmonic oscillator in the rotating wave approximation (RWA), where we neglected the terms like a^2 and $a^{\dagger 2}$, is given by

$$\mathcal{H}_{\text{int}} = g \sigma_Z a^\dagger a, \quad (12)$$

with $g = \omega_{\text{ho}} \tilde{g}^{18}$, and the external driving of the harmonic oscillator is described by

$$\mathcal{H}_{\text{drive}}(t) = f(t)(a + a^\dagger). \quad (13)$$

and throughout this work, we choose a harmonic driving force $f(t) = 2f \cos(\omega_d t)$. Neglecting the fast rotating terms $a e^{-i\omega_d t}$ and $a^\dagger e^{i\omega_d t}$, after moving in the frame rotating with frequency ω_d , the Hamiltonian becomes time independent,

$$\mathcal{H} = \mathcal{H}_S + \Delta\omega_Z a^\dagger a + f(a + a^\dagger), \quad (14)$$

with $\Delta\omega_Z = \omega_Z - \omega_d$, and the qubit-dependent frequency given by $\omega_Z = \omega_{ho}(1 + \tilde{g}\sigma_Z)$.

The qubit observable that we want to measure is $\hat{A} \equiv \sigma_Z$ and, due to the presence of the term $\Delta\sigma_X/2$, it does not represent an integral of the motion for the qubit, $[\mathcal{H}_S, \sigma_Z] \neq 0$. Therefore the measurement is not supposed to be QND, Eq. (1) not being satisfied. However, for $\Delta \ll \epsilon$ the variation in time of σ_Z becomes slow on the time scale determined by $1/\epsilon$ and one expects small deviations from an ideal QND case. The presence of the non-QND term σ_X term in \mathcal{H}_S inhibits an exact solution and a perturbative approach will be carried out in the small parameter $\Delta t \sim \Delta/\epsilon \ll 1$.

IV. SINGLE MEASUREMENT

The weak measurement of the qubit is constructed as follows. We choose the initial density matrix ($t = 0$) of the total coupled system to be the product state $\rho(0) = \rho_0 \otimes |\hat{0}\rangle\langle\hat{0}|$, with the qubit in the unknown initial state ρ_0 and the oscillator in the vacuum state $|\hat{0}\rangle$, and we let the qubit and the oscillator become entangled during the global time evolution. Suppose that at time t we perform a strong measurement of the flux quadrature $\hat{X} = \sigma(a + a^\dagger)/\sqrt{2}$, by projecting the oscillator on to the state $|x\rangle\langle x|$. This corresponds to the choice to measure the quadrature $\hat{X}(t) = \sigma(ae^{-i\omega_{ho}t} + a^\dagger e^{i\omega_{ho}t})/\sqrt{2}$ in the interaction picture,

$$x(t) = \text{Tr}[\hat{X}\rho(t)] = \text{Tr}[\hat{X}(t)\rho_R(t)], \quad (15)$$

$$\rho_R(t) = \mathcal{U}_R(t) \rho(0) \mathcal{U}_R^\dagger(t). \quad (16)$$

where an expression of $\mathcal{U}_R(t)$ and its derivation is given by Eq. (A5) in Appendix A. The operator $\mathcal{U}_R(t)$ describes the time-evolution of ρ in the rotating frame. The probability to detect the outcome x can then be written as

$$\begin{aligned} \text{Prob}(x, t) &= \text{Tr}[\langle x|\rho_R(t)|x\rangle] \\ &= \text{Tr}[\langle x|\mathcal{U}_R(t)|\hat{0}\rangle \rho_0 \langle\hat{0}|\mathcal{U}_R^\dagger(t)|x\rangle], \end{aligned} \quad (17)$$

where the trace is over the qubit space, and $\{|x\rangle\}$ is a basis of eigenstates of $\hat{X}(t)$. We define the operators

$$\mathbf{N}(x, t) = \langle x|\mathcal{U}_R(t)|\hat{0}\rangle, \quad (18)$$

$$\mathbf{F}(x, t) = \mathbf{N}^\dagger(x, t)\mathbf{N}(x, t), \quad (19)$$

acting on the qubit and, using the property of invariance of the trace under cyclic permutation, we write

$$\text{Prob}(x, t) = \text{Tr} \mathbf{F}(x, t)\rho(0). \quad (20)$$

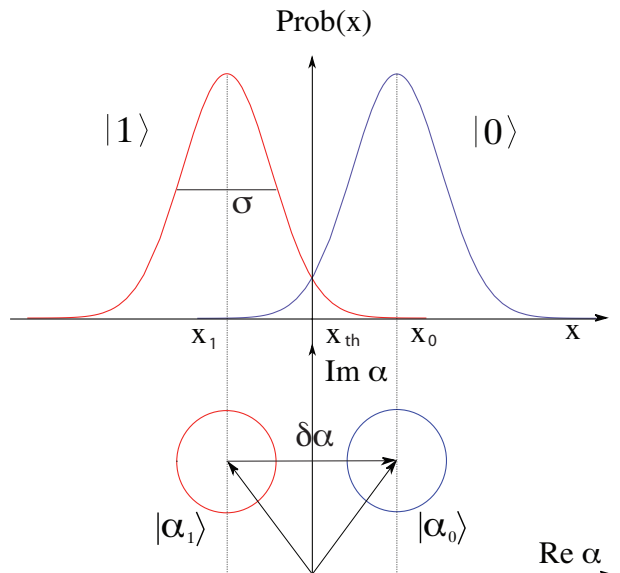


FIG. 2: Schematic description of the single measurement procedure. The coherent states $|\alpha_0\rangle$ and $|\alpha_1\rangle$, associated with the qubit states $|0\rangle$ and $|1\rangle$, are represented by a contour line of the Wigner function in the phase space, and the corresponding Gaussian probabilities are extracted.

The state of the system after the measurement is $\rho(x, t) \otimes |x\rangle\langle x|$, with the qubit in the state

$$\rho(x, t) = \frac{\mathbf{N}(x, t)\rho(0)\mathbf{N}^\dagger(x, t)}{\text{Prob}(x, t)}. \quad (21)$$

The operators $\mathbf{F}(x, t)$ are positive, trace- and hermiticity-preserving superoperators (i.e. they map density operators into density operators) acting on the qubit Hilbert space. Moreover, they satisfy the normalization condition

$$\int_{-\infty}^{\infty} dx \mathbf{F}(x, t) = \mathbf{1}, \quad (22)$$

from which the conservation of probability follows. Therefore, they form a positive operator valued measure (POVM), and we will call the operators $\mathbf{F}(x, t)$ a *continuous* POVM.

The probability distribution $\text{Prob}(x, t)$ depends strongly on the initial qubit state ρ_0 . In general $\text{Prob}(x, t)$ is expected to have a two-peak shape, arising from the two possible states of the qubit, whose relative populations determine the relative heights of the two peaks, one peak corresponding to $|0\rangle$ and the other to $|1\rangle$.

We now define an indirect qubit measurement that has two possible outcomes, corresponding to the states “0” and “1”. As a protocol for a single-shot qubit measurement, one can measure the quadrature \hat{X} and assign the state “0” or “1” to the qubit, according to the two possibilities of the outcome x to be greater or smaller

than a certain threshold value x_{th} , $x > x_{\text{th}} \rightarrow |0\rangle$ or $x < x_{\text{th}} \rightarrow |1\rangle$, as depicted in Fig. 2. Alternatively, we can infer the qubit state by repeating the procedure many times and constructing the statistical distribution of the outcome x . We then assign the relative populations of the qubit states $|0\rangle$ and $|1\rangle$ by respectively integrating the outcome distribution in the regions $\eta(1) = (x_{\text{th}}, \infty)$, $\eta(-1) = (-\infty, x_{\text{th}})$.

We formally condensate the two procedures and define a two-outcome POVM, that describes the two possible qubit outcomes, by writing

$$\mathbf{F}(s, t) = \int_{\eta(s)} dx \mathbf{F}(x, t), \quad (23)$$

$$\text{Prob}(s, t) = \text{Tr}[\mathbf{F}(s, t)\rho(0)], \quad (24)$$

with $s = \pm 1$. We will call $\mathbf{F}(s, t)$ a *discrete* POVM, in contrast to the continuous POVM $\mathbf{F}(x, t)$ defined above. Here, we introduce a convention that assigns $s = +1$ to the “0” qubit state and $s = -1$ to the “1” qubit state. The probabilities $\text{Prob}(s, t)$ are therefore obtained by integration of $\text{Prob}(x, t)$ on the subsets $\eta(s)$, $\text{Prob}(s, t) = \int_{\eta(s)} dx \text{Prob}(x, t)$. On the other hand, the probability distribution $\text{Prob}(x, t)$ is normalized on the whole space of outcomes which leads to $P(0, t) + P(1, t) = 1$ at all times. Typically, it is not possible to have a perfect mapping of the qubit state. The question is: how good can a single shot qubit measurement be?

V. EFFECTS OF THE TUNNELING σ_X TERM

Deviations from an ideal QND measurement can arise due to the presence of a non-zero σ_X term in the qubit Hamiltonian. In SC flux qubits, such a term is usually present; it represents the amplitude for tunneling through the barrier that separates the two wells of minimum potential, where the lowest energy qubit current states are located. This term cannot be switched off easily.

We can expand the full evolution operator $\mathcal{U}_R(t)$ in powers of Δt , as in Eq. (A12), and obtain a formally exact expansion of $\mathbf{F}(x, t)$,

$$\mathbf{F}(x, t) = \sum_{n=0}^{\infty} \mathbf{F}^{(n)}(x, t). \quad (25)$$

Due to the transverse ($X \perp Z$) character of the perturbation it follows that the even terms in this series (corresponding to even powers of Δt) have zero off-diagonal entries, whereas the odd terms have zero diagonal entries. Due to the normalization condition Eq. (22), valid at all orders in Δt , it can be shown that

$$\int dx \mathbf{F}^{(n)}(x, t) = \delta_{n,0} \mathbb{1}, \quad (26)$$

and consequently,

$$\sum_{s=\pm 1} \mathbf{F}^{(n)}(s, t) = \delta_{n,0} \mathbb{1}. \quad (27)$$

As a result, the probability $\text{Prob}(s, t)$ is given as a power expansion in the perturbation

$$\text{Prob}(s, t) = \sum_{n=0}^{\infty} \text{Prob}^{(n)}(s, t), \quad (28)$$

where $\sum_{s=\pm 1} \text{Prob}^{(n)}(s, t) = \delta_{n,0}$.

The expansion of the evolution operator and consequently of the continuous and discrete POVMs is in the parameter Δt . The requirement that the deviations introduced by the tunneling σ_X term in the time-evolution behave as perturbative corrections sets a time scale for the validity of the approximation, namely $t \ll 1/\Delta$, for which we will truncate the expansion up to second order. The tunneling non-QND term is considered as a perturbation in that experimentally one has $\Delta/\epsilon \ll 1$. It turns out to be convenient to choose as a time scale for the qubit measurement $t \sim 1/\epsilon$, for which follows $\Delta t \sim \Delta/\epsilon \ll 1$.

VI. TWO SUBSEQUENT MEASUREMENTS

A QND measurement implies that repeated measurements give the same result with certainty. In order to verify such a property of the measurement, we construct here the formalism that will allow us to study the correlations between subsequent measurements and make comparison with experiments^{8,9}.

After the oscillator quadrature is measured in the first step at time t and the quadrature value x is detected, the total system composed of the qubit and the oscillator is left in the state $\rho(x, t) \otimes |x\rangle\langle x|$. Such a state of the oscillator is quite unphysical, it has infinite energy and infinite indeterminacy of the $\hat{P} = (a - a^\dagger)/\sqrt{2}i$ quadrature. More realistically, what would happen in an experiment is that the oscillator is projected on to a small set of quadrature states centered around x . Moreover, after the first measurement is performed, the total system is left alone under the effects of dissipation affecting the oscillator. The reduced density matrix of a harmonic oscillator initially in a coherent state (we will see that it is actually the case) evolves, under weak coupling to a bath of harmonic oscillators in thermal equilibrium, to a mixture of coherent states with a Gaussian distribution centered around the vacuum state (zero amplitude coherent state) with variance $n_{\text{th}} = (\exp(\hbar\omega/k_B T) - 1)^{-1}$, ω being the frequency of the harmonic oscillator, T the temperature, and k_B the Boltzmann constant, whereas in the case $T = 0$ it evolves coherently to the vacuum $|\hat{0}\rangle$ ²⁰. We therefore assume that the state of the total system (qubit and oscillator) before the second measurement is

$$\rho(x, t) \otimes |\hat{0}\rangle\langle \hat{0}|. \quad (29)$$

Following the previously described procedure for the qubit single-measurement, a second measurement of the

quadrature \hat{X} with outcome y performed at time t' , having detected x at time t , would yield the conditional probability distribution

$$\text{Prob}(y, t'|x, t) = \text{Tr}[\mathbf{F}(y, t')\boldsymbol{\rho}(x, t)]. \quad (30)$$

Defining the continuous POVM qubit operators for two measurements as

$$\mathbf{F}(y, t'; x, t) = \mathbf{N}^\dagger(x, t)\mathbf{F}(y, t' - t)\mathbf{N}(x, t), \quad (31)$$

the joint probability distributions for two subsequent measurements is

$$\text{Prob}(y, t'; x, t) = \text{Prob}(y, t'|x, t)\text{Prob}(x, t) \quad (32)$$

$$= \text{Tr}[\mathbf{F}(y, t'; x, t)\boldsymbol{\rho}_0]. \quad (33)$$

The operators $\mathbf{F}(y, t'; x, t)$ satisfy the normalization condition $\int dx \int dy \mathbf{F}(y, t'; x, t) = \mathbb{1}$, ensuring the normalization of the probability distribution $\int dx \int dy \text{Prob}(y, t'; x, t) = 1$. By inspection of Eqs. (22) and (31), it follows that

$$\int dy \mathbf{F}(y, t'; x, t) = \mathbf{F}(x, t), \quad (34)$$

and the marginal distribution for the first measurement is

$$\text{Prob}_M(x, t) \equiv \int dy \text{Prob}(y, t'; x, t) = \text{Tr}[\mathbf{F}(x, t)\boldsymbol{\rho}_0], \quad (35)$$

stating that the probability to detect x in the first measurement is independent on whatever could be detected in the second measurement. On the other hand, the marginal probability distribution for the second measurement turns out to be

$$\text{Prob}_M(y, t') \equiv \int dx \text{Prob}(y, t'; x, t) = \text{Tr}[\mathbf{F}(y, t' - t)\boldsymbol{\rho}(t)], \quad (36)$$

where $\boldsymbol{\rho}(t) = \text{Tr}_S[\mathcal{U}_R(t)\boldsymbol{\rho}_0 \otimes |\hat{0}\rangle\langle\hat{0}| \mathcal{U}_R^\dagger(t)]$ is the qubit reduced density matrix at time t . We define the discrete POVM for the correlated outcome measurements as

$$\mathbf{F}(s', t'; s, t) = \int_{\eta(s)} dx \int_{\eta(s')} dy \mathbf{F}(y, t'; x, t). \quad (37)$$

Analogously to Eq. (34) it follows that $\mathbf{F}(s, t) = \sum_{s'} \mathbf{F}(s', t'; s, t)$, and the probability distribution for the outcomes of the two subsequent measurement is simply given by

$$\text{Prob}(s', t'; s, t) = \text{Tr}[\mathbf{F}(s't'; s, t)\boldsymbol{\rho}_0], \quad (38)$$

and it follows that $\sum_{s'} \text{Prob}(s', t'; s, t) = \text{Prob}(s, t) = \text{Tr}[\mathbf{F}(s, t)\boldsymbol{\rho}_0]$. The conditional probability to obtain a certain outcome s' at time t' , having obtained s at time t , is given by

$$\text{Prob}(s', t'|s, t) = \frac{\text{Tr}[\mathbf{F}(s't'; s, t)\boldsymbol{\rho}_0]}{\text{Tr}[\mathbf{F}(s, t)\boldsymbol{\rho}_0]}. \quad (39)$$

The discrete POVM for the double measurement can be in general written as

$$\mathbf{F}(s', t'; s, t) = \frac{1}{2} [\mathbf{F}(s', t')\mathbf{F}(s, t) + h.c.] + \mathbf{C}(s', t'; s, t), \quad (40)$$

where we have symmetrized the product of the two single-measurement discrete POVM operators $\mathbf{F}(s', t')$ and $\mathbf{F}(s, t)$ in order to preserve the hermiticity of each of the two terms of Eq. (40).

Proceeding as for the case of a single qubit measurement, we expand $\mathbf{F}(y, t'; x, t)$ in powers of Δ/ϵ . Equating all the equal powers of Δ/ϵ in the expansion it follows that

$$\mathbf{F}^{(n)}(s, t) = \sum_{s'} \mathbf{F}^{(n)}(s', t'; s, t), \quad (41)$$

with $\sum_{ss'} \mathbf{F}^{(n)}(s't'; s, t) = \delta_{n,0}\mathbb{1}$.

VII. IDEAL SINGLE MEASUREMENT

The case where the qubit tunneling is absent, $\Delta = 0$, satisfies the QND conditions. A single measurement alone cannot give information on the QND character of the measurement, but we can nevertheless ask the question how good a single measurement can be?

The dynamics governed by $\mathcal{U}_R^{(0)}(t)$ produces a coherent state of the oscillator, whose amplitude depends on the qubit state, see Fig. 2. In this case the continuous POVM operators have the simple form $\mathbf{F}^{(0)}(x, t) = \langle \alpha_Z(t)|x\rangle\langle x|\alpha_Z(t)\rangle$, defined through Eq. (A10) in the Appendix A. In the σ_Z -diagonal basis $\{|i\rangle\}$, with $i = 0, 1$, it is given by

$$\mathbf{F}^{(0)}(x, t)_{ij} = \delta_{ij}\mathcal{G}(x - x_i(t)), \quad (42)$$

where $x_i(t) = \sqrt{2}\sigma\text{Re}[\alpha_i(t)]$, and $\mathcal{G}(x)$ is a Gaussian of width defined by Eq. (C7). The probability distribution for the \hat{X} quadrature outcomes is thus given by

$$\text{Prob}(x, t) = \rho(0)_{00}\mathcal{G}(x - x_0(t)) + \rho(0)_{11}\mathcal{G}(x - x_1(t)). \quad (43)$$

Choosing $x_{\text{th}}(t) = (x_0(t) + x_1(t))/2$, the discrete POVM for the qubit measurement becomes

$$\mathbf{F}^{(0)}(s, t) = \frac{1}{2} \left[1 + s \text{erf} \left(\frac{\delta x(t)}{\sigma} \right) \sigma_Z \right], \quad (44)$$

where $s = \pm 1$ labels the two possible measurement outcomes, and

$$\delta x(t) = (x_0(t) - x_1(t))/2 = \frac{\sigma}{\sqrt{2}}\text{Re} \delta\alpha(t), \quad (45)$$

where $\delta\alpha(t) = \alpha_0(t) - \alpha_1(t)$, see Fig. 2. The indirect qubit measurement gives the outcome probability

$$\text{Prob}(s, t) = \frac{1}{2} \left[1 + s \text{erf} \left(\frac{\delta x(t)}{\sigma} \right) \langle \sigma_Z \rangle_0 \right], \quad (46)$$

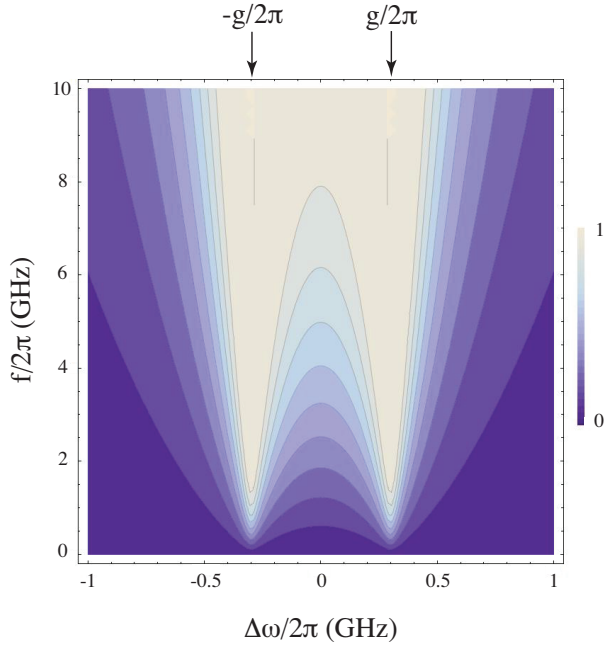


FIG. 3: $\text{Prob}(0, t = 0.1 \text{ ns})$ for the initial state $|0\rangle\langle 0|$, as given by Eq. (48), plotted as a function of the detuning $\Delta\omega/2\pi$ and the driving amplitude $f/2\pi$, for $\kappa/2\pi = 0.1 \text{ GHz}$, $g/2\pi = 0.3 \text{ GHz}$.

with $\langle \sigma_Z \rangle_0 = \text{Tr}[\sigma_Z \rho_0]$. Supposing that the qubit is prepared in the $|0\rangle$ state, one expects to find $\text{Prob}(0) = 1$ and $\text{Prob}(1) = 0$. From Eq. (46), we see that even for $\Delta = 0$ this is not always the case.

A. Short time

We choose a time $t \approx 1/\epsilon$ and a driving frequency close to the bare harmonic oscillator frequency. We can then expand the qubit dependent signal and obtain the short time behavior of the signal difference $\delta\alpha(t)$

$$\begin{aligned} \delta\alpha(t) &\approx t [A_0 e^{i\phi_0} (i\Delta\omega_0 + \kappa/2) \\ &\quad - A_1 e^{i\phi_1} (i\Delta\omega_1 + \kappa/2)] \\ &\equiv \sqrt{2} t A, \end{aligned} \quad (47)$$

where A_i , ϕ_i , $\Delta\omega_i$ and κ are given in Appendix C. The first non-zero contribution is linear in t , because the signal is due to the time-dependent driving. We measure a rotated quadrature $\hat{X}_\varphi = \sigma(ae^{-i\varphi} + a^\dagger e^{i\varphi})/\sqrt{2}$, and choose the phase of the local oscillator such that $\varphi = \arg A$. With this choice we have $\delta x(t) = \sigma|A|t$, and the probabilities for the two measurement outcomes

$$\text{Prob}(s, t) = \frac{1}{2} [1 + s \langle \sigma_Z \rangle_0 \text{erf}(|A|t)]. \quad (48)$$

By inspection of Eqs. (C9, C10) it is clear that, for driving at resonance with the bare harmonic oscillator frequency ω_{ho} , the state of the qubit is encoded in the phase

of the signal, with $\phi_1 = -\phi_0$, and $A_0 = A_1$, whereas, when matching one of the two frequencies ω_i the qubit state is encoded in the amplitude of the signal.

In Fig. 3 we plot the probability of measuring the “0” state $\text{Prob}(0, t = 0.1 \text{ ns})$ as a function of the detuning $\Delta\omega = \omega_{\text{ho}} - \omega_d$ and the driving amplitude f , given that the initial state is “0”, $\rho_0 = |0\rangle\langle 0|$. It is possible to identify a region of values of f and $\Delta\omega$ where $\text{erf}(|A|t) \approx 1$. It then follows that

$$\text{Prob}(s) \approx \frac{1}{2} [1 + s \langle \sigma_Z \rangle_0]. \quad (49)$$

This probability represents a projective measurement, for which the outcome probabilities are either 0 or 1, thus realizing a good qubit single measurement. Driving away from resonance can give rise to significant deviation from 0 and 1 to the outcome probability, therefore resulting in an imprecise mapping between qubit state and measurement outcomes and a “bad” qubit measurement.

VIII. FIRST ORDER IN TUNNELING

In order to compute the correction at first order in the tunneling term proportional to Δ we expand the evolution operator $\mathcal{U}_R(t)$ up to first order in Δt . We define the operator $\mathbf{\Pi}_Z(x, t) = \mathcal{U}_R^{(0)\dagger}(t)|x\rangle\langle x|\mathcal{U}_R^{(0)}(t)$. The contribution at first order to the continuous POVM is given by

$$\mathbf{F}^{(1)}(x, t) = -i \int_0^t dt' \langle \hat{0} | [\mathbf{\Pi}_Z(x, t), V_I(t')] | \hat{0} \rangle. \quad (50)$$

By making use of the expression Eq. (A13) for the perturbation in the interaction picture, the off-diagonal element of the first order correction to $\mathbf{F}(x, t)$ is given by

$$\begin{aligned} F^{(1)}(x, t)_{01} &= -i \frac{\Delta}{2} \int_0^t dt' \left[\mathcal{G}(x - x_0(t) + \delta x_+^{(1)}(t')) \right. \\ &\quad \left. - \mathcal{G}(x - x_1(t) - \delta x_-^{(1)}(t')^*) \right] e^{i\epsilon t'} \Gamma(t'), \end{aligned} \quad (51)$$

where the complex displacement $\delta x_{(1)}(t')$ and the overlap $\Gamma(t) \equiv \langle \alpha_0(t) | \alpha_1(t) \rangle$ are given by

$$\delta x_s^{(1)}(t') = \frac{\sigma}{\sqrt{2}} \delta\alpha(t') e^{-isg(t-t')}, \quad (52)$$

$$\Gamma(t) = \exp\left(-\frac{1}{2} |\delta\alpha(t)|^2 - i \text{Im}[\alpha_0^*(t) \alpha_1(t)]\right). \quad (53)$$

Here the state “0” is labeled by its σ_Z -eigenvalue $s = 1$, whereas the state “1” by its σ_Z -eigenvalue $s = -1$.

Analogously to the unperturbed case, the first order contribution to the discrete POVM is obtained by integrating the continuous POVM in x over the subsets $\eta(s)$. Defining the function

$$F^{(1)}(t) = i \frac{\Delta}{2} \int_0^t dt' e^{i\epsilon t'} \Gamma(t') \text{erf}\left(\frac{\delta x(t) - \delta x_+^{(1)}(t')}{\sigma}\right), \quad (54)$$

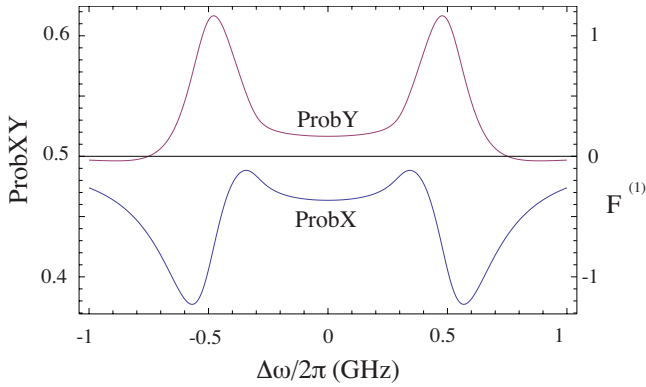


FIG. 4: (Color online) Probability ProbX and ProbY to detect the outcome $s = 1$, corrected respectively by the real and imaginary part of $F^{(1)}$, for corresponding to the initial states $|+\rangle_X\langle+|$ and $|+\rangle_Y\langle+|$, plotted versus the detuning $\Delta\omega/2\pi$. We choose $\Delta t = \Delta/\epsilon = 0.1$, $f/2\pi = 8$ GHz, $\kappa/2\pi = 0.1$ GHz, $\epsilon/2\pi = 10$ GHz, and $g/2\pi = 0.3$.

we can write the first order contribution to the discrete POVM as

$$\mathbf{F}^{(1)}(s, t) = s \left(F^{(1)}(t)|0\rangle\langle 1| + F^{(1)}(t)^*|1\rangle\langle 0| \right), \quad (55)$$

and the resulting first order correction to the probability is given by

$$\text{Prob}^{(1)}(s, t) = 2s \text{Re} \left[F^{(1)}(t)\rho(0)_{01} \right]. \quad (56)$$

This correction is valid only for short time, $t \ll 1/\Delta$. For times comparable with $1/\Delta$ a perturbative expansion of the time evolution operator is not valid. Choosing $t \approx 1/\epsilon$, we can effectively approximate

$$\delta x_s^{(1)}(t') \approx \frac{\sigma}{\sqrt{2}} \delta \alpha(t'), \quad (57)$$

and the expression for $F^{(1)}(t)$ further simplifies,

$$F^{(1)}(t) = i \frac{\Delta}{2} \int_0^t dt' e^{i\epsilon t'} - \frac{1}{2}|A|^2 t'^2 - i\psi t'^2 \text{erf}(|A|(t-t')), \quad (58)$$

with ψ given by Eq. (C11). In the ideal case $\Delta = 0$ a good measurement is achieved if $\text{erf}(|A|t) \approx 1$. Therefore we study the behavior of $F^{(1)}(t)$ in the range of driving amplitudes and frequencies that ensure a good QND measurement for the case $\Delta = 0$.

The real and imaginary part of $F^{(1)}(t)$ represent the first order correction to the outcome probability of the measurement for two particular initial states, respectively $|+\rangle_X\langle+|$ and $|+\rangle_Y\langle+|$, with $|\pm\rangle_X = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|\pm\rangle_Y = (|0\rangle \pm i|1\rangle)/\sqrt{2}$. In the first case we have

$$\text{Prob}(s, t) = \frac{1}{2} + s \text{Re}F^{(1)}(t), \quad (59)$$

and analogously for the second case, with the imaginary part instead of the real one. We see that the probability

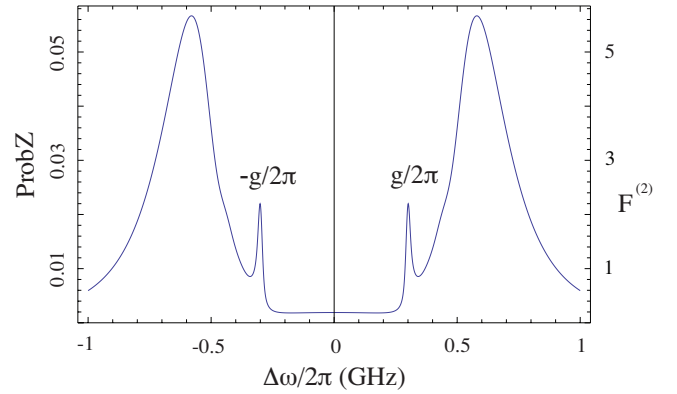


FIG. 5: Plot of the ProbZ(1) to detect "1" for the initial state $|0\rangle\langle 0|$, for $\Delta t]\Delta/\epsilon = 0.1$ as a function of the detuning $\Delta\omega/2\pi$, for $f/2\pi = 8$ GHz, $\kappa/2\pi = 0.1$ GHz, $\epsilon = 10$ GHz, and $g/2\pi = 0.3$.

to obtain "0" is increased by $\text{Re}F^{(1)}(t)$ and the probability to obtain "1" is decreased by the same amount. Since the contribution to first order in Δt only affects the off-diagonal elements of ρ_0 , there is no effect, at first order for the qubit basis states $|0\rangle$ and $|1\rangle$.

In Fig. 4 we plot the probability to detect the outcome state "0", corresponding to the outcome $s = 1$, corrected up to first order in the perturbation for $\Delta t = \Delta/\epsilon = 0.1$, for the initial states $\rho_0 = |+\rangle_X\langle+|$, that involves $\text{Re} F^{(1)}(t)$, and for the initial states $\rho_0 = |+\rangle_Y\langle+|$, that involves $\text{Im} F^{(1)}(t)$. The driving amplitude was chosen to be $f/2\pi = 8$ GHz. The two curves present a two-peak structure. As an effect of damping proportional to κ , the centers of the peaks deviate from the qubit-shifted frequencies of the harmonic oscillator. Away from these resonances they give no significant contribution to the outcome probability.

IX. SECOND ORDER IN TUNNELING

First order effects in the tunneling cannot be responsible for qubit flip during the measurement. In order to estimate the deviation from a perfect QND measurement for the eigenstates of σ_Z , we have to consider the effect of the perturbation at second order. We define $F^{(2)}(t)$ in Eq. (D7) and the contribution at second order in Δt to the discrete POVM is then

$$\mathbf{F}^{(2)}(s, t) = -sF^{(2)}(t) (|0\rangle\langle 0| - |1\rangle\langle 1|). \quad (60)$$

The dependence on s factorizes, as expected from the symmetry between the states $|0\rangle$ and $|1\rangle$, in the picture we consider with no relaxation mechanism. The correction at second order in Δ/ϵ to the outcomes probability is given by

$$\text{Prob}^{(2)}(s, t) = -sF^{(2)}(t) [\rho_{00}(0) - \rho_{11}(0)]. \quad (61)$$

In Fig. 5 we plot the second order correction to the probability to obtain "1" having prepared the qubit in the initial state ρ_0 , corresponding to $F^{(2)}(t)$, for $\Delta t = \Delta/\epsilon = 0.1$. We choose the driving amplitude $f/2\pi = 8$ GHz. The probability has a four-peak structure, two peaks at the qubit-shifted frequencies, and two peaks shifted away by the damping.

X. QND CHARACTER OF THE QUBIT MEASUREMENT

As explained in Sec. II, repeated measurements should give the same result if the measurement is QND. Such a requirement means that if a measurement projects the system onto an eigenstate of the measured observable, then a subsequent measurement should give the same result with certainty. The presence of a term that does not satisfy the QND condition may affect the character of the measurement essentially in two ways: i) by introducing deviations from the projection character of the single measurement, and ii) by generating non-zero commutators in the two-measurement POVM. These may strongly affect the two-outcome probabilities.

A. $\Delta = 0$ case

The case $\Delta = 0$ satisfies the requirement for a QND measurement of the qubit observable σ_Z . The discrete POVM factorizes in this particular case, by virtue of the fact that $[\mathbf{N}^{(0)}(y, t' - t), \mathbf{N}^{(0)}(x, t)] = 0$,

$$\mathbf{F}^{(0)}(s', t'; s, t) = \mathbf{F}^{(0)}(s', t' - t)\mathbf{F}^{(0)}(s, t). \quad (62)$$

The joint probability for the double measurement is given by

$$\text{Prob}(s', t'; s, t) = \text{Tr} \left[\mathbf{F}^{(0)}(s', t' - t)\mathbf{F}^{(0)}(s, t)\rho_0 \right]. \quad (63)$$

Choosing, e.g., $t' = 2t$, the joint probability for the two measurements reads

$$\text{Prob}(s'; s) = \frac{1}{4} \left[1 + s's \text{erf} \left(\frac{\delta x(t)}{\sigma} \right)^2 + (s' + s)\text{erf} \left(\frac{\delta x(t)}{\sigma} \right) \langle \sigma_Z \rangle_0 \right]. \quad (64)$$

In the region of driving frequency and amplitude that ensure $\text{erf}(\delta x/\sigma) \approx 1$, we find $\text{Prob}(s; s) = \text{Prob}(s)$, and $\text{Prob}(-s; s) = 0$. The conditional probability is

$$\text{Prob}(s'|s) = \frac{1 + s's + (s' + s)\langle \sigma_Z \rangle_0}{2(1 + s\langle \sigma_Z \rangle_0)}, \quad (65)$$

for which $\text{Prob}(s|s) = 1$, and $\text{Prob}(-s|s) = 0$, regardless of $\langle \sigma_Z \rangle_0$. However, it has to be noticed that in the case the condition $\text{erf}(\delta x/\sigma) \approx 1$ does not perfectly hold, the

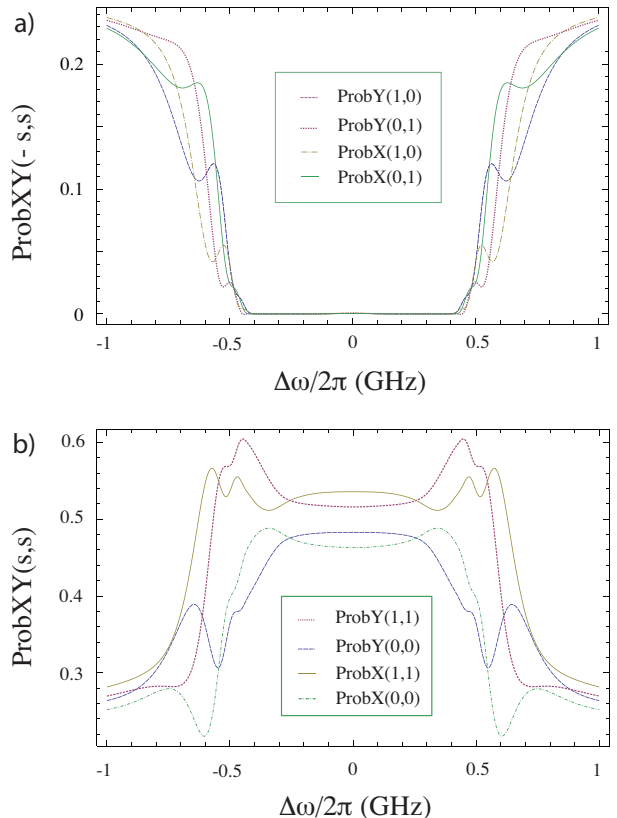


FIG. 6: Probability to obtain (a) different and (b) same outcomes in the two subsequent measurements for the initial state $|+\rangle_X |+\rangle$ and $|+\rangle_Y |+\rangle$, corrected up to first order in the perturbation, for the choice $\Delta t = \Delta/\epsilon = 0.1$, plotted versus the detuning $\Delta\omega/2\pi$. For the evaluation we set $f/2\pi = 8$ GHz, $\kappa/2\pi = 0.1$ GHz, $\epsilon = 10$ GHz, and $g/2\pi = 0.3$.

conditional probability for the two measurement to give the same outcome becomes

$$\text{Prob}(s|s) = \frac{1 + \text{erf}(\delta x/\sigma)^2 + 2s \text{erf}(\delta x/\sigma)\langle \sigma_Z \rangle_0}{2(1 + s \text{erf}(\delta x/\sigma)\langle \sigma_Z \rangle_0)}, \quad (66)$$

and this does depend on the initial state $\langle \sigma_Z \rangle_0$.

B. First order contribution

We now apply the perturbative approach in Δt to estimate the effect of the non-QND term for the joint and the conditional probabilities. Due to the transverse nature of the perturbation, it is possible to show that all the odd terms have off-diagonal entries, whereas even ones are diagonal. At first order in Δt the off-diagonal term of the

discrete POVM is given by

$$F^{(1)}(s', t'; s, t) = \frac{s}{2}F^{(1)}(t) + \frac{s'}{2}F^{(1)}(t' - t)\Gamma(t) + \frac{s'}{2}\text{erf}\left(\frac{\delta x(t' - t)}{\sigma}\right)\frac{i\Delta}{2}\int_0^t d\tau e^{i\epsilon\tau}\Gamma(\tau). \quad (67)$$

It is useful to separate the previous expression into a single measurement contribution and a first order correlation,

$$F^{(1)}(s', t'; s, t) = \frac{s}{2}F^{(1)}(t) + \frac{s'}{2}F^{(1)}(t' - t) + s'C^{(1)}(t'; t), \quad (68)$$

$$C^{(1)}(t'; t) = \frac{1}{2}(\Gamma(t) - 1)F^{(1)}(t' - t) + \frac{i\Delta}{4}\text{erf}\left(\frac{\delta x(t' - t)}{\sigma}\right)\int_0^t d\tau e^{i\epsilon\tau}\Gamma(\tau), \quad (69)$$

For the particular choice $t' = 2t$, for which the two measurement procedures are exactly the same, the joint probability for the initial state $\rho_0 = |+\rangle_X\langle +|$ is given at first order in Δt by

$$\text{Prob}(s', s) = \frac{1}{4}\left(1 + s's \text{erf}\left(\frac{\delta x(t)}{\sigma}\right)^2\right) + \frac{1}{2}(s + s')\text{Re} F^{(1)}(t) + s' \text{Re} C^{(1)}(2t; t) \quad (70)$$

We immediately observe that the probability is not symmetric with respect to s and s' . Although the driving times are the same, something is different between the first and the second measurement. The probability to obtain different outcomes $s' = -s$ is different from zero,

$$\text{Prob}(-s, s) = \frac{1}{4}\left(1 - \text{erf}\left(\frac{\delta x(t)}{\sigma}\right)^2\right) - s \text{Re} C^{(1)}(2t; t) \quad (71)$$

An analogous result holds for the initial state $\rho_0 = |+\rangle_Y\langle +|$, with the imaginary part instead of the real one. Now, no matter the sign of $C^{(1)}$, the product $-s C^{(1)}$ is negative in one case ($s = \pm 1$). In order to ensure that probabilities are non-negative one has to choose Δt small enough such that the first order negative corrections due to $C^{(1)}$ remains smaller than the unperturbed probability. If Δt is too large, one needs to take higher orders into account which should then ensure an overall non-negative probability. In Fig. 6a we plot the probability to obtain different outcomes for the initial states $|+\rangle_X\langle +|$ and $|+\rangle_Y\langle +|$ for the choice $\Delta t = \Delta/\epsilon = 0.1$.

The probability to obtain the same outcomes also contains the contributions of the first and second measure-

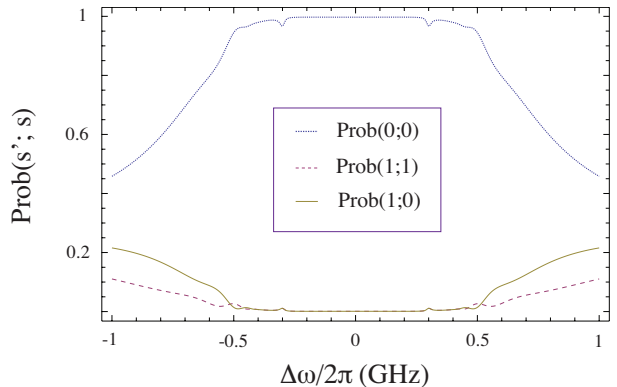


FIG. 7: Probability to obtain different ($s' = -1, s = 1$) and same outcomes ($s' = s = \pm 1$) in the two subsequent measurements for the initial state $|0\rangle\langle 0|$, corrected up to second order in the perturbation, for the choice $\Delta/\epsilon = 0.1$, plotted versus the detuning $\Delta\omega/2\pi$. For the evaluation we set $f/2\pi = 8$ GHz, $\kappa/2\pi = 0.1$ GHz, $\epsilon = 10$ GHz, and $g/2\pi = 0.3$.

ment separately,

$$\text{Prob}(s, s) = \frac{1}{4}\left(1 + s's \text{erf}\left(\frac{\delta x(t)}{\sigma}\right)^2\right) + s \text{Re} \left[F^{(1)}(t) + C^{(1)}(2t; t) \right] \quad (72)$$

By inspection of Fig. 6b we see that the probability to obtain the same outcomes for the same initial states always remains bounded between zero and one.

C. Second order contribution

The contribution to the discrete POVM at second order in Δt can be divided into a term that factorizes the contributions of the first and the second measurements, as well as a term that contains all the non-zero commutators produced in the rearrangement,

$$\mathbf{F}^{(2)}(s', t'; s, t) = \mathbf{F}^{(0)}(s, t)\mathbf{F}^{(2)}(s', t' - t) + \mathbf{F}^{(2)}(s, t)\mathbf{F}^{(0)}(s', t' - t) + \frac{1}{2}\left[\mathbf{F}^{(1)}(s, t)\mathbf{F}^{(1)}(s', t' - t) + h.c.\right] + \mathbf{C}^{(2)}(s', t'; s, t). \quad (73)$$

The full expression of the $\mathbf{C}^{(2)}$ at second order is rather involved. Choosing $t' = 2t$ we then obtain

$$C^{(2)}(p', 2t; p, t)_{ss} = p'ps C^{(2)}(t) - p'p \left| F^{(1)}(t) \right|^2, \quad (74)$$

with $C^{(2)}(t)$ given by Eq. (D8) in Appendix D. The second term Eq. (74) actually cancels the probability to have a $\pi/2$ rotation in the first measurement and a $\pi/2$ -rotation during the second.

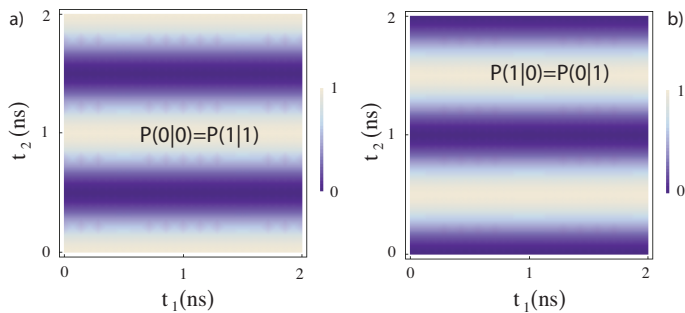


FIG. 8: Perfect QND measurement: conditional probability to obtain a) $s' = s = \pm 1$, b) $s' = -s = \pm 1$, for the case $\Delta = 0$, when rotating the qubit around the y axis at a Rabi frequency of 1 GHz before the first measurement for a time t_1 and between the first and the second measurement for a time t_2 , starting with the qubit in the state $|0\rangle\langle 0|$. The harmonic oscillator is driven at resonance with the qubit-split frequency $\Delta\omega = g$. For the evaluation we set $f/2\pi = 8$ GHz, $\kappa/2\pi = 0.1$ GHz, $\epsilon = 10$ GHz, and $g/2\pi = 0.3$.

Choosing as the initial state $\rho_0 = |0\rangle\langle 0|$, we have

$$\text{Prob}(s; s) = \frac{1}{4} \left[1 + \text{erf} \left(\frac{\delta x}{\sigma} \right)^2 + 2s \text{erf} \left(\frac{\delta x}{\sigma} \right) \right] + C^{(2)} - \left(s + \text{erf} \left(\frac{\delta x}{\sigma} \right) \right) F^{(2)}, \quad (75)$$

$$\text{Prob}(-s; s) = \frac{1}{4} \left[1 - \text{erf} \left(\frac{\delta x}{\sigma} \right)^2 \right] + F^{(2)} \text{erf} \left(\frac{\delta x}{\sigma} \right) - C^{(2)}. \quad (76)$$

The probability to obtain identical outcomes does depend on the outcome s itself, and this reflects the fact that the joint probability still depends on the initial states of the qubit. On the other hand, the probability for obtaining different outcomes does not depend on s , as expected. In Fig. 7 we plot the probability to obtain identical outcomes for the two cases $s = \pm 1$ and the probability to obtain different outcomes. We see a 5% reduction of the probability $P(0; 0)$ to detect twice in sequence the outcome "0" at frequency $\Delta\omega = \pm g$. The joint probability $\text{Prob}(s'; s)$ for two subsequent measurement up to second order in the tunneling is given by Eq. (E1) in Appendix E.

XI. RABI OSCILLATIONS BETWEEN MEASUREMENTS

In order to gain a full insight in the QND character of the measurement, we analyze the behavior of the conditional probability to detect the outcomes s and s' in two subsequent measurements when we perform a rotation of the qubit between the two measurements. Such a procedure has been experimentally adopted in the work of

Lupaşcu *et al.* [8]. When changing the qubit state between the two measurements, only partial QND behavior is expected. In addition to this, we apply an initial rotation to the qubit, such that a wide spectrum of initial states is tested. Ideally, the complete response of this procedure is supposed to be independent on the time t_1 , during which we rotate the qubit before the first measurement, and to depend only on the time t_2 , during which we rotate the qubit between the first and the second measurements, with probabilities ranging from zero to one as a function of t_2 . Such a prediction, once confirmed, would guarantee a full QND character of the measurement. In Fig. 8 we plot the conditional probabilities $P(s'|s)$ for the case $\Delta = 0$, when driving the harmonic oscillator at resonance with one of the qubit-split frequency, namely $\Delta\omega = g$. The initial qubit state is chosen to be $|0\rangle\langle 0|$. No dependence on t_1 appears and the outcomes s and s' play a symmetric role. This is indeed what we expect from a perfect QND measurement. In Fig. 9, we plot the four combinations of conditional probability $P(s'|s)$ up to second order corrections in $\Delta t = \Delta/\epsilon = 0.1$, for the choice $\Delta\omega = 0$, that is at resonance with the bare harmonic frequency. The initial qubit state is $|0\rangle\langle 0|$ and a small phenomenological relaxation time $T_1 = 10$ ns is assumed. Three features appear: i) a global reduction of the visibility of the oscillations, ii) a dependence on t_1 when the qubit is completely flipped in the first rotation, and an asymmetry under exchange of the two outcomes, with an enhanced reduction of the visibility when the first measurement produces a result that is opposite with respect to the initial qubit preparation. In fact, only at these points the initial rotation shows up. The dependence on t_1 is due to imperfections in the mapping between the qubit state and the harmonic oscillator state already at the level of the single measurement. In particular deviations from $\text{erf}(\delta x) \approx 1$ appear in the conditional probability already for the case $\Delta = 0$ for different choices of the driving frequency. The manifestation of the non-QND term comes as a reduction of the visibility of the oscillations and asymmetry of the outcomes under exchange. This is clearly shown by comparison of Fig. 8 and Fig. 9. The combined effect of the quantum fluctuations of the oscillator together with the tunneling between the qubit states is therefore responsible for deviation from a perfect QND behavior, although a major role is played, as expected, by the non-QND tunneling term.

XII. CONCLUSION

In this paper we have analyzed the QND character of a qubit measurement based on coupling to a harmonic oscillator that works as a pointer to the qubit states. The Hamiltonian that describes the interaction between the qubit and the oscillator does not commute with the qubit Hamiltonian. This would in principle inhibit a QND measurement of the qubit. The term in the qubit Hamilto-

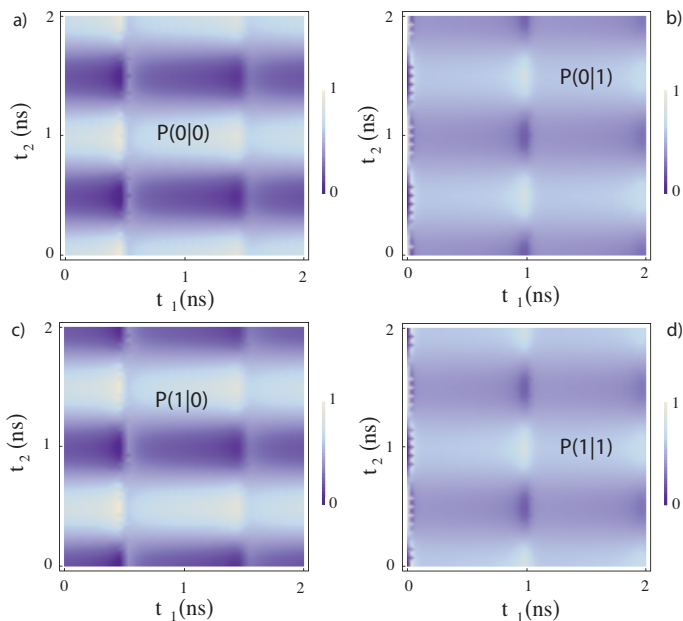


FIG. 9: Conditional probability to obtain a) $s' = s = 1$, b) $s' = -s = 1$, c) $s' = -s = -1$, and d) $s' = s = -1$ for the case $\Delta t = \Delta/\epsilon = 0.1$, when rotating the qubit around the y axis before the first measurement for a time ωt_1 and between the first and the second measurement for a time ωt_2 , starting with the qubit in the state $|0\rangle\langle 0|$. Correction in Δt are up to second order. The harmonic oscillator is driven at resonance with the bare harmonic frequency. For the evaluation we set $f/2\pi = 8$ GHz, $\kappa/2\pi = 0.1$ GHz, $\epsilon = 10$ GHz, and $g/2\pi = 0.3$.

nian that gives rise to the non-zero commutator is small compared with the qubit energy gap, and in the short time qubit dynamics it can be viewed as a small perturbation. The perturbative analysis carried out for fast measurements leads us to the conclusion that the effect of the non-QND term can manifest itself as a non negligible correction. A perfect QND measurement guarantees perfect correlations in the outcomes of two subsequent measurements, therefore QND character of the measurement is understood in terms of deviations from the expected behavior. Corrections to the outcome probabilities have been calculated up to second order in the perturbing term.

The ground state and the excited state of the qubit are affected only at second order by the perturbation, but a general measurement protocol should prescind from the state being measured. Therefore, in the spirit of the experiment of Lupaşcu *et al.* [8], we have studied the conditional probability for the outcomes of two subsequent measurements when rotating the qubit before the first measurement and between the first and the second measurement. In the case where the QND condition is perfectly satisfied, that is when the perturbation is switched off, no dependence of the conditional probability on the duration of the first rotation appears, whereas Rabi oscil-

lations between the two measurement range from zero to one. This behavior shows perfect QND character of the qubit measurement. On the other hand, the main effects of the non-QND term manifests as an overall reduction of the visibility of the oscillations and a asymmetry between the outcomes of the measurements. An additional dependence on the duration of the first qubit rotation may appear in case a projective measurement of the qubit is not achieved already in absence of the perturbing non-QND term. Experimentally this effect has not been observed yet, which might be because relaxation processes inhibit a perfect flip of the qubit before the first measurement. Indeed, addition of a phenomenological qubit relaxation rate hides a dependence on the first rotation duration.

We point out that our analysis is valid only when the non-QND term $\Delta\sigma_X$ can be viewed as a perturbation, that is for short time $\Delta t \ll 1$ and when the qubit dynamics is dominated by the term $\epsilon\sigma_Z$, for $\Delta/\epsilon \ll 1$. Our analysis is not valid for the case $\epsilon = 0$. In the present study we have neglected the non-linear character of the SQUID, which is not relevant to the fundamental issue described here, but plays an important role in some measurement procedures⁶⁻⁹.

A way to improve the QND efficiency would be simply to switch the tunneling off. In the case of superconducting flux qubit, a possibility toward smaller Δ could be to gate the superconducting islands between the junctions of the qubit loop [23]. As an operational scheme one could think of working at finite Δ for logical operations and then at $\Delta = 0$ for the measurement.

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Appendix A: Exactly solvable case: $\Delta = 0$

In order to determine the evolution governed by the Hamiltonian Eq. (14) we single out the term \mathcal{H}_0 diagonal in the $\{|s, n\rangle\}$ basis, with $|s\rangle$ the eigenstates of σ_Z and $|n\rangle$ the oscillator Fock states,

$$\mathcal{H} = \mathcal{H}_0 + f(a + a^\dagger) + \frac{\Delta}{2}\sigma_X, \quad (\text{A1})$$

with $\mathcal{H}_0 = \epsilon\sigma_Z/2 + \Delta\omega_Z a^\dagger a$. We then work in the interaction picture with respect to \mathcal{H}_0 . The Heisenberg equation for the density operator reads $\dot{\rho}_I = -i[\mathcal{H}_I, \rho_I]$, with

$$\mathcal{H}_I = \mathcal{H}_I^{(0)} + V_I, \quad (\text{A2})$$

$$\mathcal{H}_I^{(0)} = f(ae^{-i\Delta\omega_Z t} + a^\dagger e^{i\Delta\omega_Z t}), \quad (\text{A3})$$

$$V_I = \frac{\Delta}{2} \left(e^{i\hat{\Omega}_n t} \sigma_+ + e^{-i\hat{\Omega}_n t} \sigma_- \right), \quad (\text{A4})$$

where we define $\hat{\Omega}_n = \epsilon + 2ga^\dagger a$, and $\sigma_\pm = (\sigma_X \pm i\sigma_Y)/2$. We will call $\mathcal{U}_I(t)$ the evolution operator generated by \mathcal{H}_I .

The evolution operator is given by $\mathcal{U}(t) = \exp(-i\omega_d a^\dagger a - i\mathcal{H}_0 t)\mathcal{U}_I(t)$. For the measurement procedure so far defined we are interested in the evolution operator in the frame rotation at the bare harmonic oscillator frequency. Therefore

$$\mathcal{U}_R(t) = \exp(-i\epsilon t \sigma_Z / 2 - i\mathcal{H}_{\text{int}} t)\mathcal{U}_I(t). \quad (\text{A5})$$

For the case $\Delta = 0$ the model is exactly solvable and $\mathcal{U}_I^{(0)}(t)$ can be computed as shown in the Appendix B via a generalization of the Baker-Hausdorff formula¹⁹,

$$\mathcal{U}_I^{(0)}(t) = D(\gamma_Z(t)), \quad (\text{A6})$$

where $D(\alpha) = \exp(a^\dagger \alpha - a \alpha^*)$ is a displacement operator²⁰, and

$$\gamma_Z(t) = -if \int_0^t ds e^{i\Delta \omega_Z}. \quad (\text{A7})$$

In the frame rotating at the bare harmonic oscillator frequency, the state of the oscillator is a coherent state whose amplitude depends on the qubit state. A general initial state

$$\rho_{\text{tot}}(0) = \sum_{ij=0,1} \rho_{ij} |i\rangle\langle j| \otimes |\hat{0}\rangle\langle \hat{0}|, \quad (\text{A8})$$

where $|\hat{0}\rangle$ is the harmonic oscillator vacuum state, evolves to

$$\rho_R(t) = \sum_{ij=0,1} \rho_{ij} |i\rangle\langle j| \otimes |\alpha_i(t)\rangle\langle \alpha_j(t)|, \quad (\text{A9})$$

where we define the qubit operators $\alpha_Z(t) \equiv \gamma_Z(t)e^{-igt\sigma_Z}$, and the object

$$|\alpha_Z(t)\rangle \equiv D(\alpha_Z)e^{-i\mathcal{H}_{\text{int}}t}|\hat{0}\rangle, \quad (\text{A10})$$

that gives a qubit-dependent coherent state of the harmonic oscillator, once the expectation value on a qubit state is taken, $|\alpha_i(t)\rangle = \langle i|\alpha_Z(t)\rangle|i\rangle$, for $i = 0, 1$.

1. Perturbation theory in Δ

For non-zero Δ , a formally exact solution can be written as

$$\mathcal{U}_I(t) = \mathcal{U}_I^{(0)}(t)\mathcal{T} \exp\left(-i\Delta \int_0^t dt' \mathcal{V}_I(t')\right), \quad (\text{A11})$$

with $\mathcal{V}_I(t) = \mathcal{U}_I^{(0)\dagger}(t)\mathcal{V}_I(t)\mathcal{U}_I^{(0)}(t)$ and \mathcal{T} the time order operator. For a time scale $t \ll 1/\Delta$ we expand the evolution operator in powers of $\Delta t \ll 1$,

$$\begin{aligned} \mathcal{U}_I(t) \approx & \mathcal{U}_I^{(0)}(t) \left(\mathbb{1} - i\Delta t \int_0^1 d\tau \mathcal{V}_I(\tau t) \right. \\ & \left. - (\Delta t)^2 \int_0^1 d\tau \int_0^\tau d\tau' \mathcal{V}_I(\tau t)\mathcal{V}_I(\tau' t) \right). \end{aligned} \quad (\text{A12})$$

The interaction picture potential can be written as

$$\mathcal{V}_I(t) = \frac{1}{2} [\mathcal{D}(t)\sigma^+ + \mathcal{D}^\dagger(t)\sigma^-], \quad (\text{A13})$$

with the oscillator operators $\mathcal{D}(t)$ defined as

$$\mathcal{D}(t) = D^\dagger(\gamma_0(t))e^{i\Omega_n t}D(\gamma_1(t)) \quad (\text{A14})$$

$$\begin{aligned} &= \exp(i\epsilon t - i\text{Im}[\alpha_0(t)\alpha_1(t)^*]) \\ &\times D(-\delta\alpha(t)e^{igt})e^{2igt a^\dagger a}. \end{aligned} \quad (\text{A15})$$

Here $\delta\alpha(t) = \alpha_0(t) - \alpha_1(t)$ is the difference between the amplitudes of the coherent states associated with the two possible qubit states.

Appendix B: Evolution Operator

It is possible to show¹⁹ that, given two time dependent operators $A(t)$ and $B(t)$ that satisfy $[A(t), A(t')] = [B(t), B(t')] = 0$ and $[A(t), B(t')] = f(t, t')$, such that $f(t, t')$ commutes with $A(t'')$ and $B(t'')$ for all t, t', t'' , a solution to

$$\dot{U}(t) = (A(t) + B(t))U(t), \quad (\text{B1})$$

with $U(0) = 1$ is provided by

$$\begin{aligned} U(t) = & \exp\left[\int_0^t ds (A(s) + B(s))\right] \\ & \times \exp\left[-\frac{1}{2} \int_0^t \int_0^t ds ds' \text{sign}(s - s') f(s, s')\right] \end{aligned} \quad (\text{B2})$$

that provides a generalization of the Baker-Hausdorff formula. The time dependent phase characterizing the qubit evolution due to the driving of the harmonic oscillator can be neglected for a time $t \sim 1/\epsilon$, for which $g/\epsilon \ll 1$.

Appendix C: Quadratures and coherent states

The quadratures of a harmonic oscillator (or field mode), \hat{X} and \hat{P} , are given in terms of the respective creation and annihilation operators a and a^\dagger as

$$\hat{X} = \sigma\sqrt{\frac{\hbar}{2}}(a + a^\dagger), \quad (\text{C1})$$

$$\hat{P} = -\frac{i}{\sigma}\sqrt{\frac{\hbar}{2}}(a - a^\dagger). \quad (\text{C2})$$

Such a definition implies that from $[a, a^\dagger] = 1$ follows that $[\hat{X}, \hat{P}] = i\hbar$, and the Hamiltonian is

$$H = \frac{\omega}{2} \left(\sigma^2 \hat{P}^2 + \frac{\hat{X}^2}{\sigma^2} \right) = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right). \quad (\text{C3})$$

The normalized coherent state wave function is

$$\langle x|\alpha\rangle = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\left[\frac{\alpha}{2}(\alpha - \alpha^*) - \frac{1}{2\sigma^2}(x - \sqrt{2}\sigma\alpha)^2\right], \quad (\text{C4})$$

and the coherent state overlap is found to be

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right]. \quad (\text{C5})$$

In the main text we also made extensive use of the following relation

$$\langle\alpha|x\rangle\langle x|\beta\rangle = \langle\alpha|\beta\rangle\mathcal{G}\left(x - \frac{\sigma}{\sqrt{2}}(\alpha^* + \beta)\right) \quad (\text{C6})$$

where $\mathcal{G}(x)$ is a Gaussian

$$\mathcal{G}(z) = \exp(-z^2/\sigma^2)/(\sigma\sqrt{\pi}). \quad (\text{C7})$$

Introducing a rate κ that describes the Markovian damping of the harmonic oscillator by a zero-temperature bath of harmonic oscillators, the coherent state qubit-dependent amplitude $\alpha_i(t)$ is found to be²¹

$$\alpha_i(t) = A_i e^{i\phi_i} \left[1 - e^{-i\Delta\omega_i t - \kappa t/2}\right], \quad (\text{C8})$$

with $\Delta\omega_i = \omega_i - \omega_d$. The qubit-dependent amplitude is

$$A_i = \frac{f}{\sqrt{(\omega_i - \omega_d)^2 + \kappa^2/4}}, \quad (\text{C9})$$

and the qubit-dependent phase is

$$\phi_i = \arctan\left(\frac{\omega_i - \omega_d}{\kappa/2}\right) - \frac{\pi}{2}. \quad (\text{C10})$$

Through these quantities we can approximate the phase associated with two different coherent states as $\text{Im}[\alpha_0(t)\alpha_1(t)^*] \approx \psi t^2$, with ψ given by

$$\psi = \text{Im}\left[A_0 A_1 e^{i(\phi_0 - \phi_1)} (i\Delta\omega_0 + \kappa/2)(-i\Delta\omega_1 + \kappa/2)\right]. \quad (\text{C11})$$

Appendix D: Second order quantities $F^{(2)}$ and $C^{(2)}$

For time $t \approx 1/\epsilon$, we expand the evolution operator in Δt and collect the contributions that arise at second power in (Δ/ϵ) ,

$$\mathbf{F}^{(2)}(x, t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \left[\langle \hat{0} | \mathcal{V}_I(t_1) \mathbf{\Pi}_Z(t, x) \mathcal{V}_I(t_2) | \hat{0} \rangle - \langle \hat{0} | \mathbf{\Pi}_Z(t, x) \mathcal{V}_I(t_1) \mathcal{V}_I(t_2) | \hat{0} \rangle \right] + h.c. \quad (\text{D1})$$

The first term arises from the linear expansion of the evolution operator acting on the left and the right of the oscillator projection operator $|x\rangle\langle x|$, and it represents the

sum of the contributions at time $t_1, t_2 < t$ of the perturbation acting on the backward-in-time evolved projection operator $\mathbf{\Pi}_Z(x, t)$, whereas the second term arises from the quadratic expansion acting on the left or on the right of $\mathbf{\Pi}_Z(x, t)$.

By making use of the expression Eq. (A13) for the perturbation in the interaction picture, the qubit components of the second order contribution to the continuous POVM can be written as

$$F^{(2)}(x, t)_{ss} = \frac{\epsilon^2}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 \text{Re} \left\{ \mathcal{O}_s(t_1, t_2) \times \left[\mathcal{G}\left(x - x_{\bar{s}}(t) - \bar{s} \xi_{\bar{s}}^{(2)}(t_1, t_2)\right) - \mathcal{G}\left(x - x_s(t) - s \zeta_s^{(2)}(t_1, t_2)\right) \right] \right\} \quad (\text{D2})$$

where $\bar{s} = -s$,

$$\mathcal{O}_s(t_1, t_2) = \exp\left(is\epsilon(t_1 - t_2) - is\psi(t_1^2 - t_2^2)\right) \times \langle \delta\alpha(t_1) e^{isgt_1} | \delta\alpha(t_2) e^{isgt_2} \rangle, \quad (\text{D3})$$

and $\langle\alpha|\beta\rangle$ is the overlap between coherent states, as given by Eq. (C4), and

$$\xi_s^{(2)}(t_1, t_2) = \delta x_s^{(1)}(t_1)^* + \delta x_s^{(1)}(t_2), \quad (\text{D4})$$

$$\zeta_s^{(2)}(t_1, t_2) = -\delta x_s^{(1)}(t_1) + \delta x_s^{(1)}(t_2). \quad (\text{D5})$$

The first term $\xi_s^{(2)}(t_1, t_2)$ represents the complex displacement of the oscillator position due to the perturbation acting one time at $t_2 < t$ (forward in time), and one time at $-t_1 > -t$ (backward in time). The second term $\zeta_s^{(2)}(t_1, t_2)$ represents the displacement of the oscillator due to the perturbation acting two times at $t_2 < t_1 < t$. Between the two perturbations the system evolves freely for the time $t_2 - t_1$ and accumulates a phase that depends on the difference of the effective qubit-dependent frequencies. In the short time approximation $t \approx 1/\epsilon$ such a phase can be neglected. Integrating the position degree of freedom over the subsets $\eta(s')$, we obtain

$$F^{(2)}(s', t)_{ss} = -s's \frac{\epsilon^2}{4} \int_0^t dt_1 \int_0^{t_1} dt_2 \text{Re} \left\{ \mathcal{O}_s(t_1, t_2) \times \left[\text{erf}\left(\frac{\delta x(t) + \xi_{\bar{s}}^{(2)}(t_1, t_2)}{\sigma}\right) + \text{erf}\left(\frac{\delta x(t) + \zeta_s^{(2)}(t_1, t_2)}{\sigma}\right) \right] \right\}. \quad (\text{D6})$$

Now, the same considerations that lead to Eq. (57) apply also here and we can simplify the integrand. The correction $F^{(2)}(t)$ at second order to the discrete POVM is evaluated to be

$$F^{(2)}(t) = \frac{\epsilon^2}{4} \int_0^t dt_1 \int_0^{t_1} dt_2 \cos(\epsilon(t_1 - t_2) - \psi(t_1^2 - t_2^2)) \times e^{-\frac{1}{2}|A|^2(t_1 - t_2)^2} \left(\text{erf}(|A|(t + t_1 + t_2)) + \text{erf}(|A|(t - t_1 + t_2)) \right). \quad (\text{D7})$$

The full expression of the diagonal matrix element of the second order contribution $C^{(2)}$ is

$$\begin{aligned}
C^{(2)}(t) &= \frac{\epsilon^2}{8} \int_0^t dt_1 \int_0^t dt_2 e^{i(\epsilon(t_1-t_2)-\psi(t_1^2-t_2^2))} \\
&\times e^{-\frac{|A|^2}{2}(t_1-t_2)^2} \text{erf}(|A|(t+t_1+t_2)) \\
&- \text{Im} \left[F^{(1)}(t) \epsilon \int_0^t dt' e^{-i\epsilon t'} \Gamma(t) \Gamma(t')^* \right. \\
&\times \left. e^{-\frac{1}{2}|At'|^2+|A|^2 t' t} \text{erf} \left(\frac{\delta x_-^{(1)}(t')}{\sigma} \right) \right]. \quad (\text{D8})
\end{aligned}$$

Appendix E: Joint probability for two subsequent measurements up second order in the tunneling term

All in all the joint probability up to second order in Δt to obtain the outcomes s for the first measurement and s' for the second one, given an unknown qubit initial state ρ_0 , is

$$\begin{aligned}
\text{Prob}(s'; s) &= \frac{1}{4} \left[1 + s's \text{erf} \left(\frac{\delta x(t)}{\sigma} \right)^2 + (s' + s) \text{erf} \left(\frac{\delta x(t)}{\sigma} \right) \langle \sigma_Z \rangle_0 \right] + \frac{1}{2} (s + s') \left[\langle \sigma_X \rangle_0 \text{Re} F^{(1)}(t) - \langle \sigma_Y \rangle_0 \text{Im} F^{(1)}(t) \right] \\
&+ s' \left(\langle \sigma_X \rangle_0 \text{Re} C^{(1)}(t) - \langle \sigma_Y \rangle_0 \text{Im} C^{(1)}(t) \right) - \left(\frac{1}{2} (s' + s) F^{(2)}(t) - s's C^{(2)}(t) \right) \langle \sigma_Z \rangle_0 \\
&- s's F^{(2)}(t) \text{erf} \left(\frac{\delta x(t)}{\sigma} \right). \quad (\text{E1})
\end{aligned}$$

* Electronic address: luca.chirolli@uni-konstanz.de

† Electronic address: guido.burkard@uni-konstanz.de

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