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Group theory and symmetries in quantum mechanics
Summer semester 2017 - Exercise sheet 6
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Problem 16: Reduction of representations of D_4 on restriction to C_2.

Consider the 2-dimensional irreducible representation E of the group D_4 (this group and its character table were considered in problem 11). Reduce it on restriction to the group C_2.

Problem 17: Components of the momentum operator.

Prove that the components of the momentum operator in the quantum mechanics, $p_x = -i\hbar\partial_x$, $p_y = -i\hbar\partial_y$ and $p_z = -i\hbar\partial_z$, transform according to the same representation (reducible or irreducible) as the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

Problem 18: Particle confined to a 2D-square.

Consider a particle of mass m confined to a square in two dimensions whose vertices are located at the following points of the xy-plane: (L, L), $(L, -L)$, $(-L, -L)$, and $(-L, L)$. The potential is assumed to be zero within the square and infinitely large at the edges of the square. The stationary eigenfunctions of the corresponding quantum-mechanical problem have the form:

$$\psi_{p,q}(x, y) = \frac{1}{L} \left\{ \begin{array}{c} \cos(k_p x) \\ \sin(k_p x) \end{array} \right\} \left\{ \begin{array}{c} \cos(k_q y) \\ \sin(k_q y) \end{array} \right\} ,$$

where $k_p = \frac{p\pi}{2L}$ and $k_q = \frac{q\pi}{2L}$, with p and q being positive integers. The notation with $\{\}$ in the equation above means that $\cos(k_p x)$ is taken if p is odd, $\sin(k_p x)$ is taken if p is even, and similarly for the other factor. The corresponding eigenvalues are

$$E_{p,q} = \frac{\hbar^2}{8mL^2}(p^2 + q^2) .$$

(a) What is the symmetry group of the Hamiltonian of this problem? Copy its character table here for the further steps.

(b) For which irreducible representations do the eigenfunctions $\psi_{1,1}(x, y)$ and $\psi_{2,2}(x, y)$ form bases?

(c) For which irreducible representation do the eigenfunctions $\psi_{1,2}(x, y)$ and $\psi_{2,1}(x, y)$ form a basis?

(d) What is the degeneracy corresponding to $(p = 6, q = 7)$ and $(p = 2, q = 9)$? Is this a normal or accidental degeneracy?

(e) Think of a modification of the system which does not destroy its symmetry but partly lifts this degeneracy.

(f) Check if all of the irreducible representations occur in the Hilbert space spanned by the eigenfunctions $\psi_{p,q}(x, y)$.