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Group theory and symmetries in quantum mechanics
Summer semester 2016 - Exercise sheet 8
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Problem 22: Mapping $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$**: part 1.**

In the following two excises we investigate the relation between the matrix groups SU(2) and SO(3). In order to establish a connection between SU(2) and SO(3) we consider the transformation

$$\mathbf{r} \longmapsto \mathbf{r}' \equiv R\mathbf{r} \quad \text{with} \quad U\mathbf{r}\cdot\boldsymbol{\sigma}U^\dagger = \mathbf{r}'\cdot\boldsymbol{\sigma} \ ,$$

where $U \in \mathrm{SU}(2)$, $\mathbf{r} \in \mathbb{R}^3$, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^{\mathrm{T}}$ denotes the vector of the Pauli matrices.

a) Show that R is a linear operator and that this operator does not change angles and distances between vectors.
Hint : Consider how the scalar product of two vectors changes under this transformation and use the property $\mathrm{Sp}(\sigma_i\sigma_j) = 2\delta_{ij}$. Here and in what follows use the Einstein notation for summation.

b) Show that in the basis of cartesian coordinates the components of R can be written as

$$R_{ij} = \frac{1}{2}\mathrm{Sp}\left(\sigma_i U \sigma_j U^\dagger\right) \ .$$

c) Using a possible parametrization of U by the components of a unit vector $\mathbf{n}$ and the angle α as $U = e^{i\alpha\mathbf{n}\cdot\boldsymbol{\sigma}}$, show that

$$R_{ij} = \cos 2\alpha\, \delta_{ij} + \sin 2\alpha\, \epsilon_{ijk} n_k + (1-\cos 2\alpha) n_i n_j \ ,$$

where we use the Einstein notation for summation.
Hint : Use $e^{i\alpha\mathbf{n}\cdot\boldsymbol{\sigma}} = \mathbb{1}\cos\alpha + i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\alpha$, $\sigma_j\sigma_k = \delta_{jk}\mathbb{1} + i\epsilon_{jkl}\sigma_l$ with ϵ_{jkl} being the Levi-Civita symbol, and $\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$.

d) Write R for $\mathbf{n} = \mathbf{e}_x$, $\mathbf{n} = \mathbf{e}_y$ and $\mathbf{n} = \mathbf{e}_z$. Which rotations do you get in these cases?

Problem 23: Mapping $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$**: part 2.**

a) Using 22c show that the matrices R belong to SO(3).
Hint : In order to prove this, you should show that the components of R are real, $R^{\mathrm{T}}R = \mathbb{1}$, and $\det R = 1$.

b) Differentiating the result of 22c with respect to α at the following points in the parameter manifold, $\{\alpha = 0, n_x = 1, n_y = n_z = 0\}, \{\alpha = 0, n_y = 1, n_x = n_z = 0\}, \{\alpha = 0, n_x = 1, n_y = n_z = 0\}$, find a possible form of three generators of SO(3).

c) Argue why the mapping $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ with $U \longmapsto R$ is a homomorphism. I.e. for each matrix $R \in \mathrm{SO}(3)$ there is at least one matrix $U \in \mathrm{SU}(2)$ generating the corresponding rotation, and arbitrary U_1, U_2 from SU(2) lead to $R(U_1U_2) = R(U_1)R(U_2)$.