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Group theory and symmetries in quantum mechanics
Summer semester 2016 - Exercise sheet 5
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Problem 14: Clebsch-Gordan coefficients for a direct product of representations.

The coordinates x, y of a particle transform according to the 2-dimensional irreducible representation E of the group D_4 (cf. problem 11), so that the products of coordinates x_1x_2, x_1y_2, y_1x_2 and y_1y_2 of two such particles transform according to the product representation $E \bigotimes E$. Find the four combinations of these product functions which transform irreducibly under D_4 (*Hint: look at the character table of D_4 and the result of problem 11d*). Derive the Clebsch-Gordan coefficients from them.

Problem 15: Reduction of representations of D_4 on restriction to C_2.

Consider the 2-dimensional irreducible representation E of the group D_4, mentioned in the previous problem. Reduce it on restriction to the group C_2.

Problem 16: Components of the momentum operator.

Prove that the components of the momentum operator in the quantum mechanics, $p_x = -i\hbar\partial_x$, $p_y = -i\hbar\partial_y$ and $p_z = -i\hbar\partial_z$, transform according to the same representation (reducible or irreducible) as the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

Problem 17: Projection operators, D_3 and quadratic polynomials of x and y.

Consider the group D_3 and the function x^2. The operations of D_3 generate an invariant subspace L from this function. A representation O of D_3 in this subspace was considered in problem 7, with functions x^2, y^2 and xy as a basis.

a) Using the character table of D_3, construct the projection operators $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ (corresponding to the irreducible representations $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ of D_3) and apply them to the function x^2. What is the dimension of the space spanned by $P^{(1)}x^2$, $P^{(2)}x^2$ and $P^{(3)}x^2$?

b) Using the matrices of the representation O and the matrices of the irreducible representations of D_3 given in the lecture course (lecture 6), project the function x^2 onto the space $L_{3,i}$ of functions transforming according to the i-th row of the representation $T^{(3)}$, i.e. calculate $P_i^{(3)}x^2$ ($i = 1, 2$).

c) Solving (b), you can see that with help of $P_i^{(3)}$ you can not generate from x^2 a complete basis of the space L_3 belonging to the representation $T^{(3)}$. Make use of the transfer operators $P_{ij}^{(3)}$ in order to obtain a complete basis in L_3 from x^2. Are the corresponding functions orthogonal or not? Explain the reason.