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Group theory and symmetries in quantum mechanics
Summer semester 2016 - Exercise sheet 4
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Problem 11: Characters and representations of the group D_4.

Consider the group D_4 of all possible (in 3D) rotations of a square. In Problem 3 we determined the classes and the multiplication table of this group. The classes can be denoted as E, C_4^2, $2C_4$, $2C_2$ and $2C_2'$ (the number in front indicates the number of elements in the corresponding class).

a) Construct the character table for this group using the orthogonality relations of the characters and the property that 1-dimensional representations coincide with their own characters.

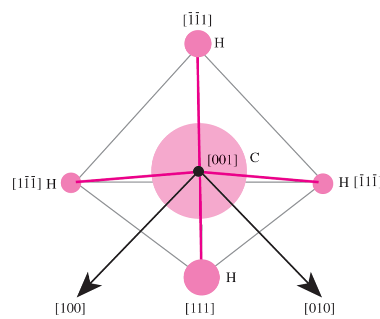
b) Select the x- and y-axes being parallel to the corresponding sides of the square and the z-axis perpendicular to the plane of the square and piercing through its centrum. Construct a 3-dimensional representation of this group using the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

c) Find the characters of the representation found in (b). Using the character table of D_4, found in (a), show that this representation reduces to a direct sum of a 2-dimensional and a 1-dimensional representation. Can you see this result directly from the matrices found in (b)?

d) Calculate the characters of the direct product of the two-dimensional irreducible representation of D_4 with itself and decompose this direct product in irreducible representations.

Problem 12: Character table for the group T_d.

Consider the symmetry group of the methane molecule (CH_4). It is the symmetry of the regular tetrahedron, called T_d. This group has 24 elements in 5 classes: E, $3C_2$, $6S_4$, 6σ, $8C_3$. Here the number in front indicates the number of elements in the corresponding class, C_n denote rotations by angles $\pm\frac{2\pi}{n}$ about some axes, σ denote reflections in respect to some planes, S_n denote rotations by angles $\pm\frac{2\pi}{n}$ about some axes combined with reflections in respect to the plane perpendicular to the corresponding rotation axis (improper rotations).



Methane molecule

a) Determine the axes for the rotations and improper rotations as well as the planes for the reflections. Check that all classes of this group are rational.

Problem 13: The regular representation

Prove that the matrices $T^R(G_a)$, $G_a \in G$ introduced in the lecture indeed form a representation of a group G.