electron states in a magnetic field

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Overwiev

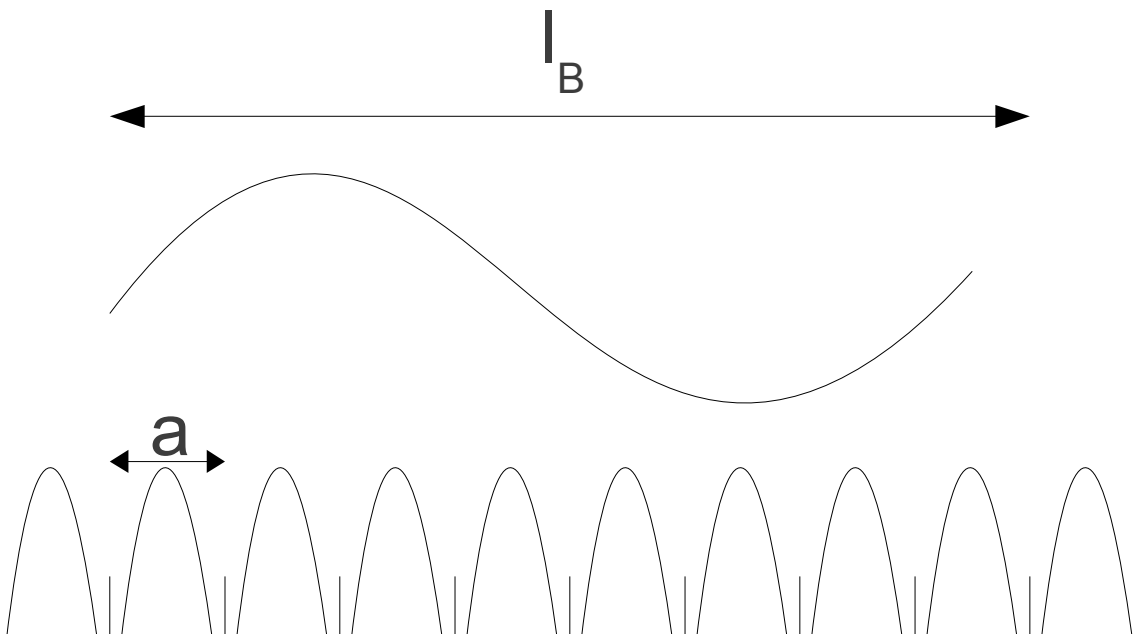
- effective Hamiltonian
  - assumptions
  - replacing $\vec{p}$ by $\vec{p} - \frac{e}{c}\vec{A}$
- monolayer graphene
  - Landau levels (energy spectra)
  - degeneracy of the eigenstates
- bilayer graphene
  - intermediate energies
  - trigonal warping
  - unified description

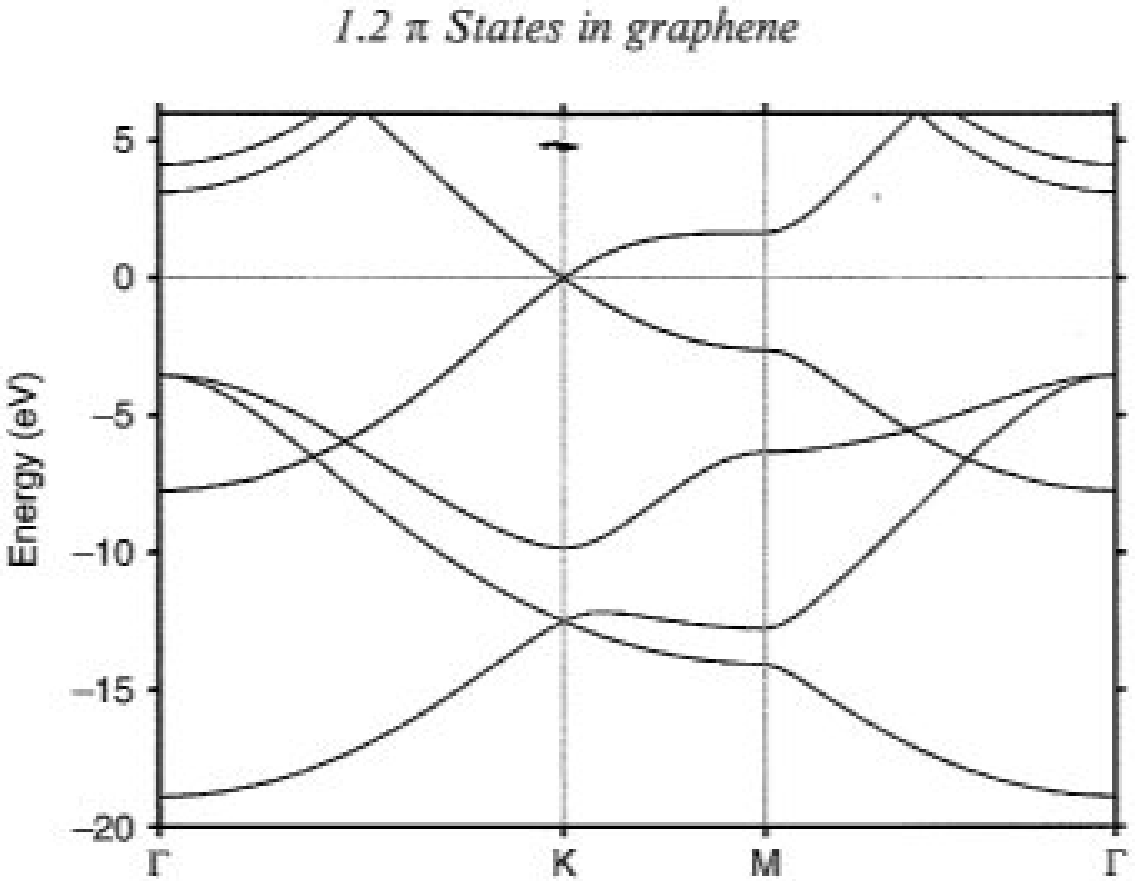
General purpose

$$\begin{array}{rcccc} \hat{H} & = & \dfrac{\vec{p}^2}{2m} + V(\vec{r}) & \rightarrow & \hat{H} = \nu\vec{\sigma}\cdot\vec{p} \\ \textit{exp.} & \downarrow & \textit{verified} & & \downarrow\,? \\ \hat{H} & = & \dfrac{\vec{\pi}^2}{2m} + V(\vec{r}) & \rightarrow & \hat{H} = \nu\vec{\sigma}\cdot\vec{\pi} \end{array}$$

$$\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$$

$$l_B = \sqrt{\frac{\hbar c}{|e|B}} \gg a$$




1.2 π States in graphene
Energy (eV)
5
0
−5
−10
−15
−20
Γ
K
M
Γ

$$\hat{H} = \frac{\vec{\pi}^2}{2m} + V(\vec{r}) \tag{1}$$

$$H\psi = E\psi \quad ; \quad \psi = \sum_i c_i \varphi_i(\vec{r}) \tag{2}$$

$$\varphi_i = \varphi_0\left(\vec{r} - \vec{R}_i\right) = \exp\left(-\frac{i}{\hbar}\vec{R}_i\hat{\vec{p}}\right)\varphi_0\left(\vec{r}\right) \tag{3}$$

$$\hat{\vec{\Pi}} = \vec{p} + \frac{e}{c}\vec{A} \quad ; \quad \hat{\vec{\pi}} = \vec{p} - \frac{e}{c}\vec{A} \tag{4}$$

$$\rightarrow \tilde{\psi} = \sum_i c_i \tilde{\varphi}_i(\vec{r}) \tag{5}$$

$$H_{ij}\left(B \neq 0\right) =$$

$$= \int d\vec{r}\, \varphi_0^*\left(\vec{r}\right) \hat{H}\left(B \neq 0\right) \exp\left(\frac{i}{\hbar}\vec{R}_i\hat{\vec{\Pi}}\right) \exp\left(-\frac{i}{\hbar}\vec{R}_j\hat{\vec{\Pi}}\right)\, \varphi_0\left(\vec{r}\right)$$

$$= \exp\left(\frac{-ie}{2\hbar c}\left(\vec{R}_i \times \vec{R}_j\right)\vec{B}\right) \cdot$$

$$\cdot \underbrace{\int d\vec{r}\, \varphi_0^*\left(\vec{r}\right) \hat{H}\left(B \neq 0\right) \exp\left(\frac{i}{\hbar}\left(\vec{R}_i - \vec{R}_j\right)\hat{\vec{p}}\right)\, \varphi_0\left(\vec{r}\right)}_{t_{ij}(\vec{\pi})}$$

$$H_{ij}\left(B = 0\right) = t_{ij}(\vec{p}) =$$

$$= \int d\vec{r}\, \varphi_0^*\left(\vec{r}\right) \hat{H}\left(B = 0\right) \exp\left(\frac{i}{\hbar}\left(\vec{R}_i - \vec{R}_j\right)\hat{\vec{p}}\right)\, \varphi_0\left(\vec{r}\right)$$

$$\hat{H}_{eff} = t_{ij}(\vec{\pi})$$

while

$$\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$$

$$[\pi_x, \pi_y] = \frac{ie\hbar B}{c}$$

$$\pi_- = \pi_x - i\pi_y \tag{6}$$

$$\pi_+ = \pi_x + i\pi_y \tag{7}$$

with

$$[\pi_-, \pi_+] = -\frac{2|e|\hbar B}{c} \tag{8}$$

$$-\frac{c}{2|e|\hbar B}\left[\pi_-, \pi_+\right] = 1 \tag{9}$$

$$\left[b, b^\dagger\right] = 1 \tag{10}$$

$$\begin{array}{ccccc} \hat{H} & = & \dfrac{\vec{p}^2}{2m} + V(\vec{r}) & \rightarrow & \hat{H} = \vec{\sigma} \cdot \vec{p} \\ exp. & \downarrow & verified & & \downarrow ? \\ \hat{H} & = & \dfrac{\vec{\pi}^2}{2m} + V(\vec{r}) & \rightarrow & \hat{H} = \vec{\sigma} \cdot \vec{\pi} \end{array}$$

In the vicinity of K:

$$\hat{H}_K = \nu \vec{\sigma} \vec{p} \tag{11}$$

$$\hat{H}_{eff} = \nu \begin{pmatrix} 0 & \hat{\pi}_- \\ \hat{\pi}_+ & 0 \end{pmatrix} = \nu \sqrt{\frac{2|e|\hbar B}{c}} \begin{pmatrix} 0 & \hat{b} \\ \hat{b}^\dagger & 0 \end{pmatrix} \tag{12}$$

$$\hat{b}\psi_2 = \varepsilon\psi_1 \tag{13}$$

$$\hat{b}^\dagger\psi_1 = \epsilon\psi_2 \tag{14}$$

$$E = \nu \sqrt{\frac{2|e|\hbar B}{c}} \varepsilon \equiv \frac{\nu\sqrt{2}\hbar}{l_B} \varepsilon \tag{15}$$

There is a zero-energy solution with:

$$\psi_1 = 0 \tag{16}$$
$$\psi_2 = |0\rangle \tag{17}$$
$$b\psi_2 = b|0\rangle = 0 \tag{18}$$
$$b^\dagger 0 = \varepsilon|0\rangle = 0 \rightarrow \varepsilon = 0 \tag{19}$$

applying $b^\dagger$ to $b\psi_2 = \varepsilon\psi_1$ we get

$$b^\dagger b\psi_2 = \varepsilon \underbrace{b^\dagger\psi_1}_{\varepsilon\psi_2} = \varepsilon^2\psi_2 \tag{20}$$

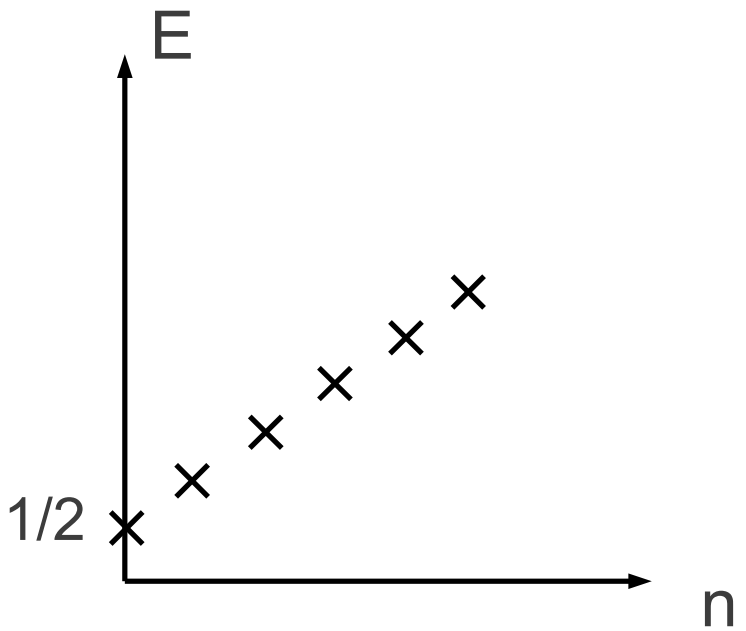
$$\rightarrow \varepsilon_n^2 = n = 0, 1, 2, 3... \tag{21}$$

$$\rightarrow \varepsilon_n^2 = n = 0, 1, 2, 3... \tag{22}$$

$$\rightarrow E_n^{(\pm)} = \pm\hbar\omega_c\sqrt{n} \tag{23}$$

$$\hbar\omega_c = \frac{\sqrt{2}\hbar\nu}{l_B} \tag{24}$$

non-relativistic-electrons

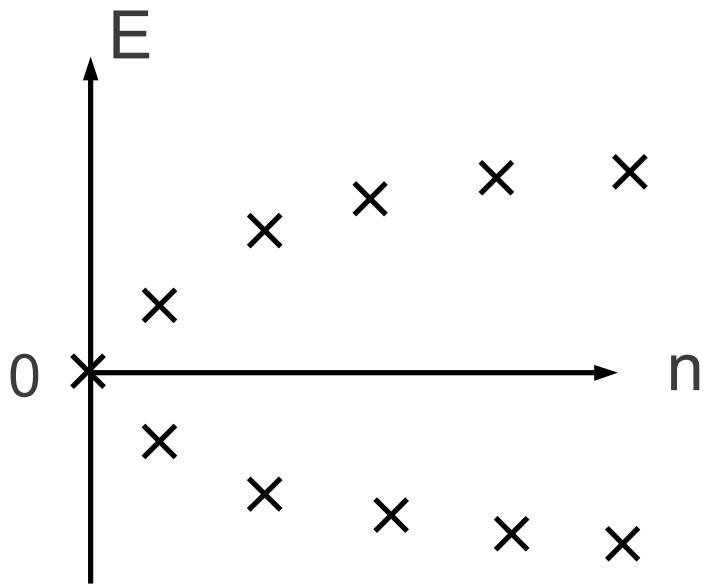


equidistant spectrum

$$\varepsilon = \hbar\omega\left(\tfrac{1}{2} + n\right)$$

$$t(\vec{\pi}) = \frac{\vec{\pi}^2}{2m^*}$$

electrons in graphene



not equidistant spectrum

$$E_n^{(\pm)} = \pm\hbar\omega_c\sqrt{n}$$

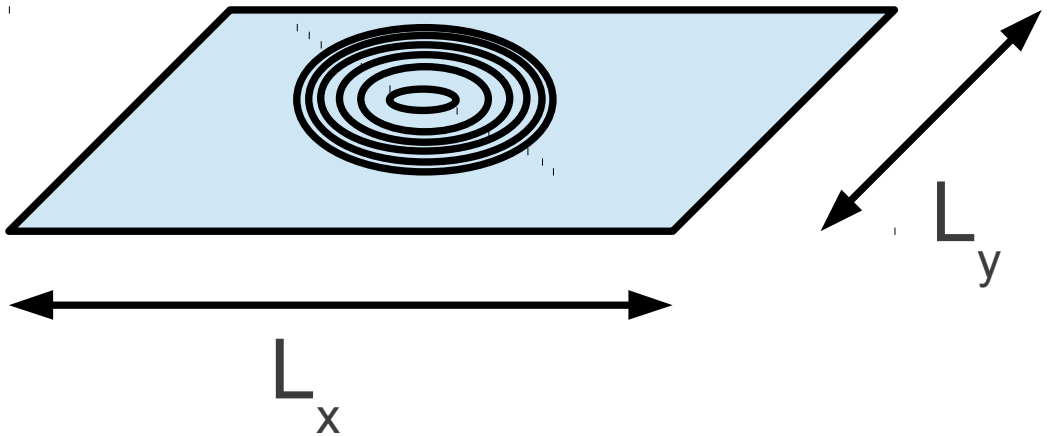
$$t(\vec{\pi}) = \begin{pmatrix} 0 & \hat{\pi}_- \\ \hat{\pi}_+ & 0 \end{pmatrix}$$

Degeneracy

$$x_0 = l_B^2 k_y$$

$$k_y L_y = 2\pi m$$

$$k_y = \frac{2\pi}{L_y} m$$



$$m = 0, 1, 2...$$

$$0 \leq x_0 = l_B^2 k_y = l_b^2 \frac{2\pi}{L_y} m \leq L_x$$

$$\rightarrow m \leq \frac{L_x L_y}{2\pi l_B^2}$$

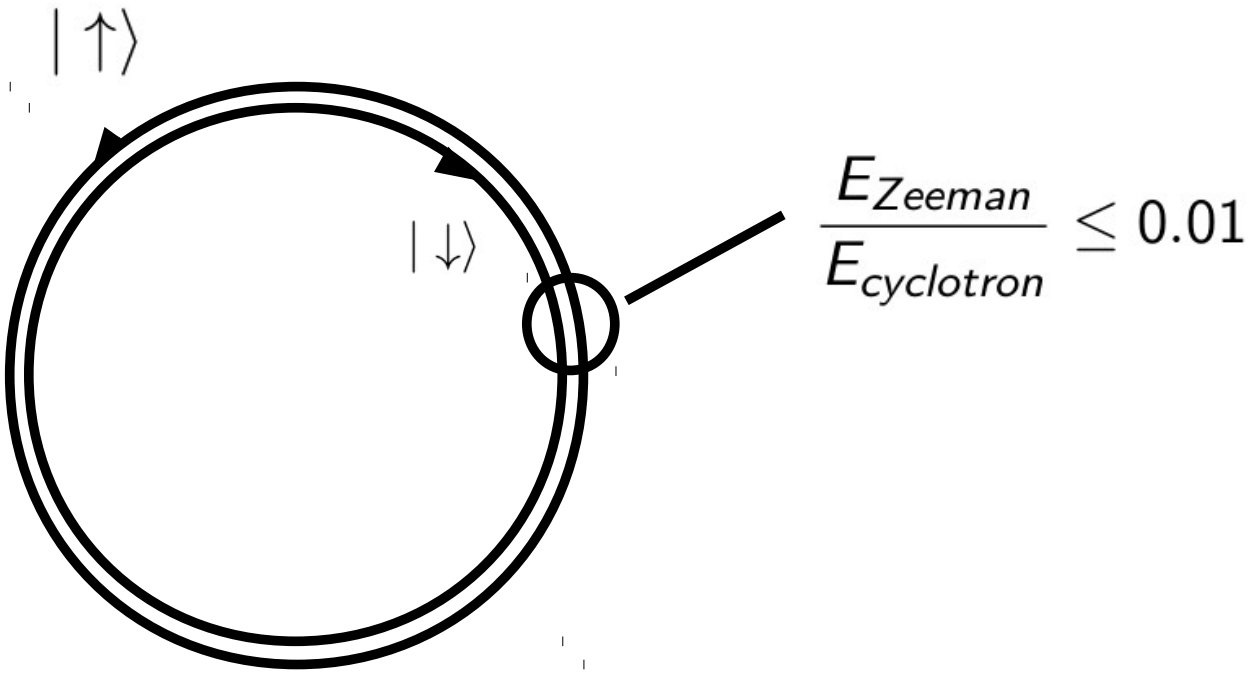
$$\rightarrow m \leq \frac{L_x L_y}{2\pi l_B^2} \quad ; \quad l_B = \sqrt{\frac{\hbar c}{|e|B}}$$

$$m_{max} = \tilde{g} = \frac{L_x L_y}{2\pi} \frac{|e|B}{\hbar c} = \frac{A}{2\pi} \frac{|e|B}{\hbar c}$$

$$\tilde{g} = \frac{A}{2\pi} \frac{|e|B}{h/(2\pi)c} = \frac{\Phi}{\Phi_0} \quad ; \quad \Phi_0 = \frac{hc}{|e|}$$

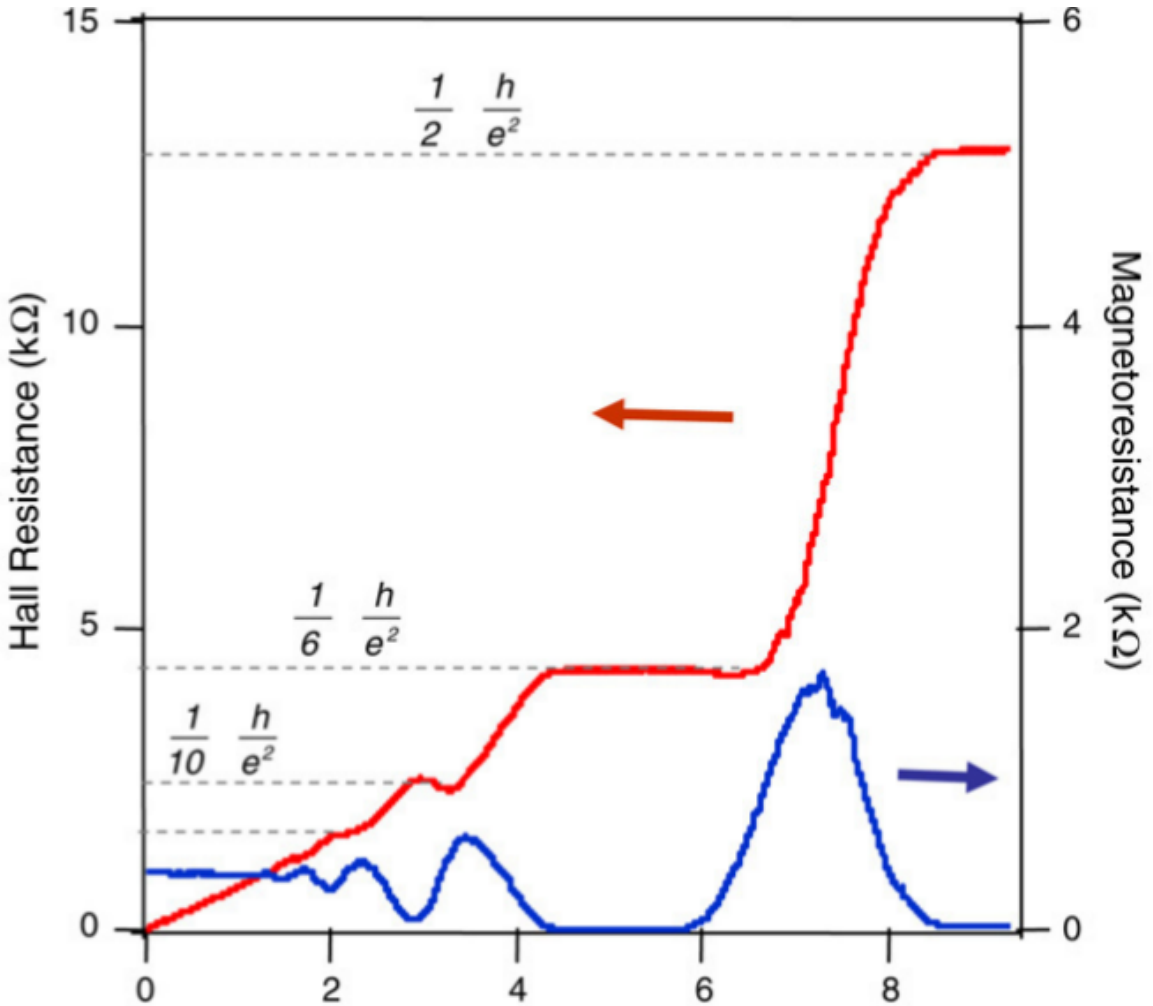
$$g = 2 \cdot 2 \cdot \tilde{g} \quad ; \quad \text{one factor for } K, K'$$

and another factor 2 for the two spin directions:

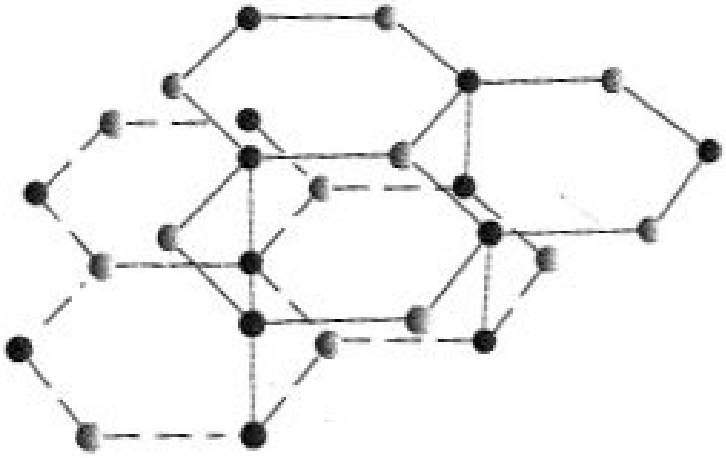


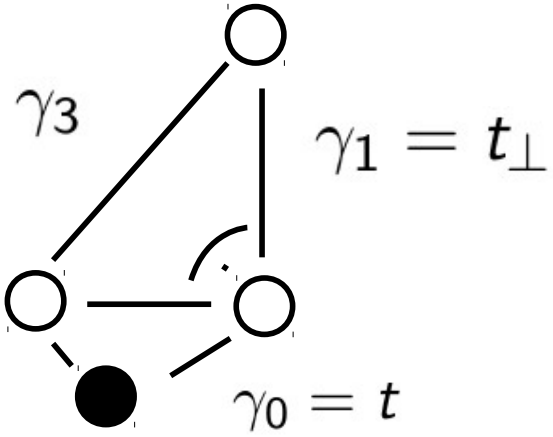
=>almost the same energy

$$R_{xy}^{-1} = \pm 4 \left(|n| + \frac{1}{2} \right) \frac{e^2}{\hbar}$$



Landau levels in bilayer graphene





3 cases:

$$|E| \approx |t_\perp| \qquad \text{high energies}$$

$$\gamma_3^2 \frac{|t_\perp|}{t^2} \ll |E| \ll |t_\perp| \qquad \text{intermediate energies}$$

$$|E| \approx \gamma_3^2 \frac{|t_\perp|}{t^2} \qquad \text{low energies}$$

intermediate energies

$$\gamma_3^2 \frac{|t_\perp|}{t^2} \ll |E| \ll |t_\perp| \qquad \text{intermediate energies}$$

$$\begin{aligned}
\hat{H} &= \frac{1}{2m^*} \begin{pmatrix} 0 & (\hat{p}_x - i\hat{p}_y)^2 \\ (\hat{p}_x + i\hat{p}_y)^2 & 0 \end{pmatrix} \\
\hat{\tilde{H}}_K &= \frac{1}{2m^*} \begin{pmatrix} 0 & \hat{\pi}_-^2 \\ \hat{\pi}_+^2 & 0 \end{pmatrix} \\
\pi_- &= \pi_x - i\pi_y = \sqrt{\frac{2|e|\hbar B}{c}}\, b \\
\pi_+ &= \pi_x + i\pi_y = \sqrt{\frac{2|e|\hbar B}{c}}\, b^\dagger
\end{aligned}$$

$$\hat{\tilde{H}}_K = \frac{1}{2m^*}\sqrt{\frac{2|e|\hbar B}{c}}^2 \begin{pmatrix} 0 & \hat{b}^2 \\ \hat{b}^{\dagger\,2} & 0 \end{pmatrix}$$

$$\omega_c^* := \frac{|e|B}{m^* c}$$

$$\hat{\tilde{H}}_K = \frac{2|e|\hbar B}{2m^* c} \begin{pmatrix} 0 & \hat{b}^2 \\ \hat{b}^{\dagger\,2} & 0 \end{pmatrix} = \hbar\omega_c^* \begin{pmatrix} 0 & \hat{b}^2 \\ \hat{b}^{\dagger\,2} & 0 \end{pmatrix}$$

	bilayer	monolayer
$b^2\psi_2$	$= \varepsilon\psi_1$	$b\psi_2 = \varepsilon\psi_1$
$(b^\dagger)^2\psi_1$	$= \varepsilon\psi_2$	$b^\dagger\psi_1 = \varepsilon\psi_2$

$$b^2|0\rangle = 0 \quad ; \quad b^2|1\rangle = 0$$

$$\begin{aligned} \rightarrow g_0 &= 2\frac{\Phi}{\Phi_0} \\ g &= \frac{\Phi}{\Phi_0} \quad \text{for} \quad m \geq 2 \end{aligned}$$

$$\left(\hat{b}\right)^2 \psi_2 = \varepsilon\psi_1$$
$$\left(\hat{b}^\dagger\right)^2 \psi_1 = \varepsilon\psi_2$$

$$\left(b^\dagger\right)^2 b^2 \psi_2 = \varepsilon^2\psi_2$$
$$b^\dagger b^\dagger b\, b\, \psi_2 = \varepsilon^2\psi_2 \quad ; \quad \left[b, b^\dagger\right] = 1$$
$$b^\dagger \left(bb^\dagger - 1\right) b = b^\dagger b \left(b^\dagger b - 1\right)$$

$$b^\dagger b \left(b^\dagger b - 1 \right) \psi_2 \;=\; \varepsilon^2 \psi_2$$

$$\varepsilon = E_\nu = \pm \hbar \omega_c^* \sqrt{n(n-1)}$$

For large $n \gg 1$:

$$\sqrt{n^2 \left(1 - \frac{1}{n} \right)} = |n| \sqrt{1 - \frac{1}{n}} \approx |n| \left(1 - \frac{1}{2n} \right)$$

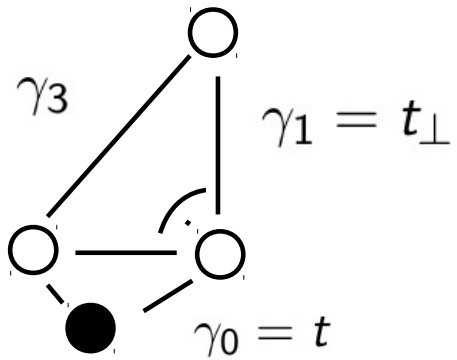
$$\rightarrow E_\nu \approx \pm \hbar \omega_c^* \left(|n| - \frac{1}{2} \right)$$

3 cases:

$$|E| \approx |t_\perp| \qquad \text{high energies}$$

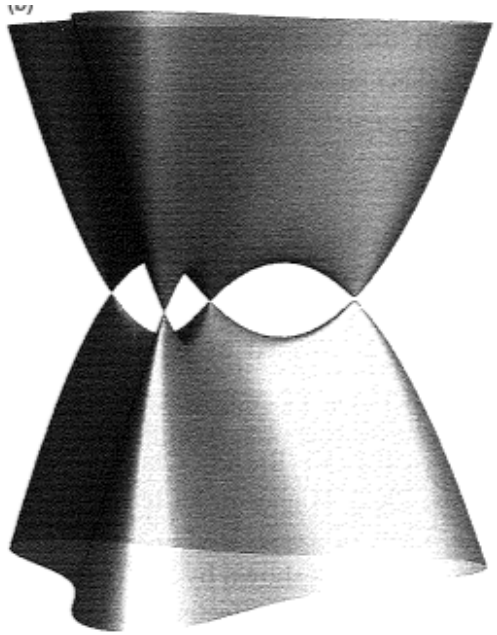
$$\gamma_3^2 \frac{|t_\perp|}{t^2} \ll |E| \ll |t_\perp| \qquad \text{intermediate energies}$$

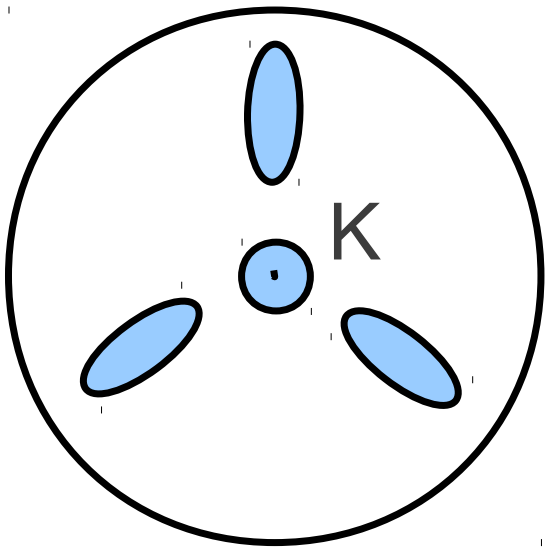
$$|E| \approx \gamma_3^2 \frac{|t_\perp|}{t^2} \qquad \text{low energies}$$

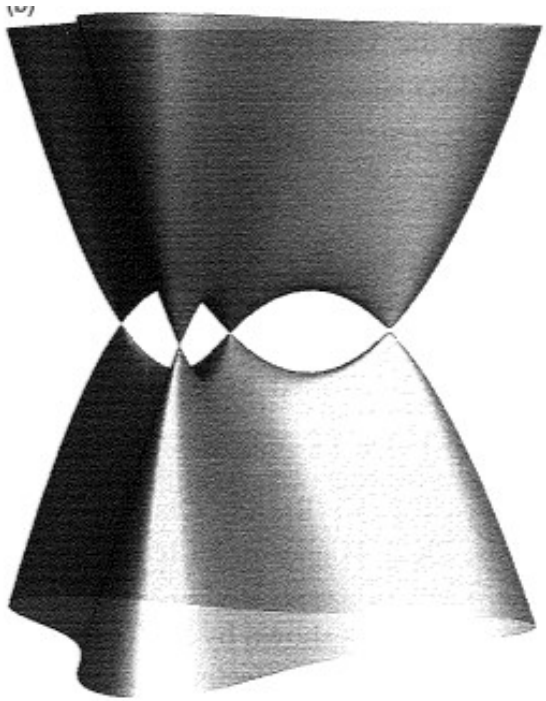


Trigonal warping (low energies)

$$
\begin{aligned}
|E| &\approx \gamma_3^2 \frac{\gamma_1}{\gamma_0^2} = \gamma_3^2 \frac{|t_\perp|}{t^2} \\
\hat{H} &= \hbar\omega_c \begin{pmatrix} 0 & b^2 + \alpha b^\dagger \\ \left(b^\dagger\right)^2 + \alpha b & 0 \end{pmatrix} \\
\alpha &= \frac{\gamma_3 a m^*}{\hbar^2} \sqrt{\frac{2\hbar c}{|e|B}} \\
\tilde{g} &= 2 \cdot 2 \cdot 4 \cdot g \qquad \text{for low enough energies}
\end{aligned}
$$



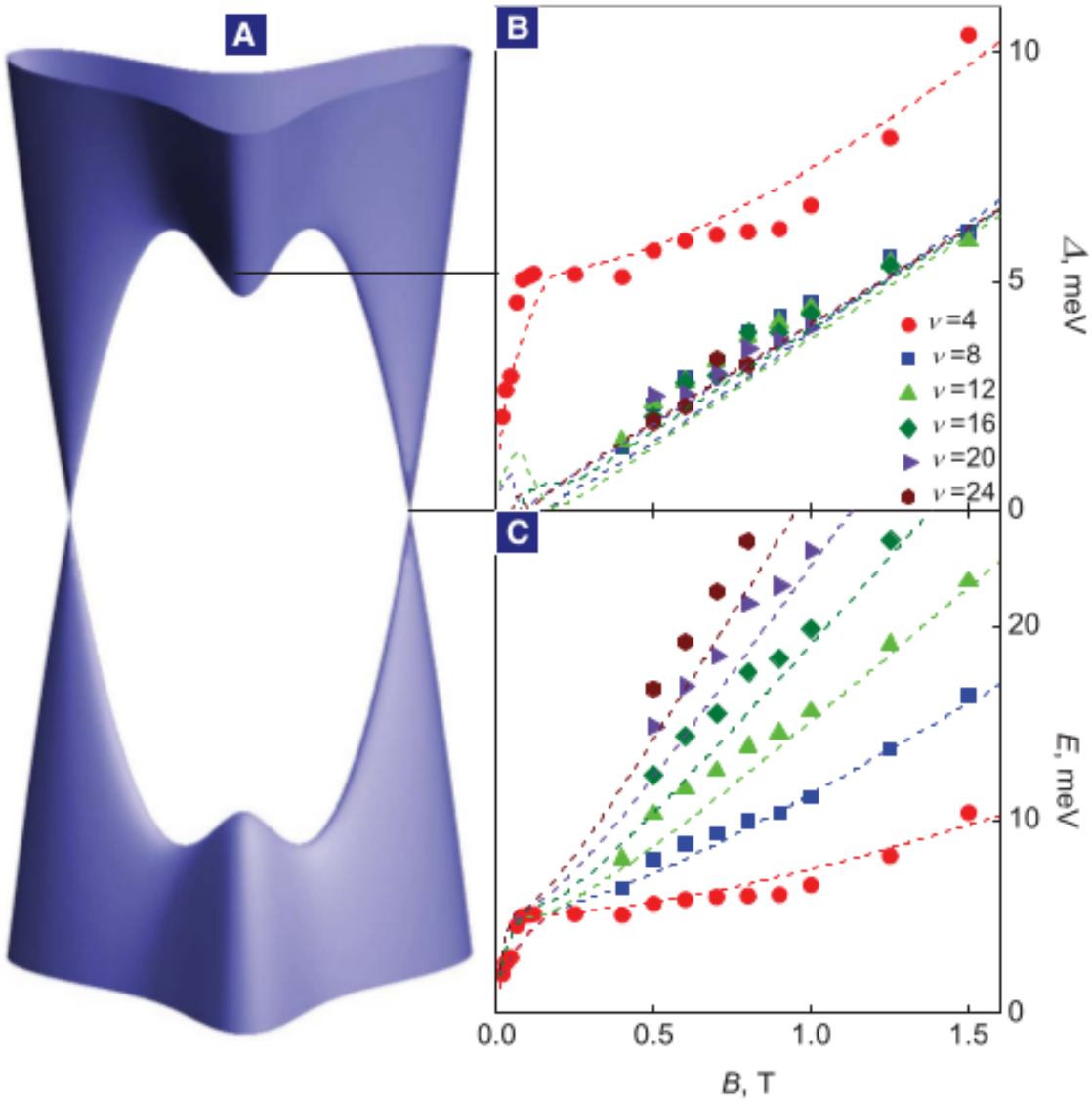




$$\tilde{g} = 16 \cdot g = 16 \cdot \frac{\Phi}{\Phi_0} \quad \text{theoretically}$$

$$\tilde{g} = 8 \cdot g \quad \text{experimentally}$$

ν : "filling factor"



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$$|E| \approx \gamma_3^2 \frac{|t_\perp|}{t^2} \qquad \text{low energies}$$

Unified description (higher energies)

$$|E| \geq |t_\perp|$$

$$\hat{H} = \begin{pmatrix} 0 & \nu\vec{\pi}_+ & t_\perp & 0 \\ & 0 & 0 & 0 \\ & & 0 & \nu\vec{\pi}_- \\ (\ldots)^\dagger & & & 0 \end{pmatrix}$$

$$t_\perp = \Gamma\sqrt{\frac{2|e|B\nu^2}{c}}$$

$$\varepsilon_n^2 = \frac{\Gamma^2 + 2n + 1}{2} \pm \sqrt{\left(\frac{\Gamma^2 + 2n + 1}{2}\right)^2 - n(n+1)}$$

For $\Gamma = 0$:

$$\varepsilon_n^2 = n + \frac{1}{2} \pm \frac{1}{2}$$

For $\Gamma \gg 1$:

$$\varepsilon_{n1}^2 = \frac{n(n+1)}{\Gamma^2}$$

$$\varepsilon_{n2}^2 = \Gamma^2 + 2n + 1$$

$$\begin{aligned} \varepsilon_{n2} &\approx \pm\left[\Gamma + \frac{1}{\Gamma}\left(n+\frac{1}{2}\right)\right] \\ \text{For } \Gamma &\approx 1 \\ B_c &\approx \frac{2}{9}\left(\frac{t_\perp}{t}\right)^2 \frac{\hbar c}{|e|a^2} \approx 70\,T \end{aligned}$$

conclusion

- effective Hamiltonian
  - assumptions
  - replacing $\vec{p}$ by $\vec{p} - \frac{e}{c}\vec{A}$
- monolayer graphene
  - Landau levels (energy spectra)
  - degeneracy of the eigenstates
- bilayer graphene
  - intermediate energies
  - trigonal warping
  - unified description

References:


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