

Quantum Hall Effect in graphene

Seminar: Electronic properties of graphene

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1 2DEG

1.1 Experimental facts in a 2DEG

A 2DEG (2 dimensional electron gas) is build mostly in a MOS structure (metall(*Al*) oxide(*SiO₂*) semiconductor(*Al*)). By adding a gate voltage V_G between the metall and the semiconductor a thin layer of quasi free electrons is build between the semiconductor and the insulator (=oxide). The gate voltage forces the conduction band of the semiconductor to bend down towards the insulator, so that it very far to the insulator under the level of the fermi energy. Because of this there exists some electron states which now can be occupied and the 2DEG is build. The thickness of this layer is only about 5-10 *nm*, so we ca treat it as a two dimensional system. The wavelength of these electrons is much larger than the lattice constant of the semiconductor so that the periodic potential can be averaged an the electrons can be treated as quasi-free.

For measurments a corrent is generated through the 2DEG (x-y-plane) in x-direction and a magnetic field penetrates the 2DEG perpendicular to the current. We measure the resistivity and the conductivity in x- and y-direction. We call the conductivity in x-direction the diagonal conductivity σ_{xx} , and in y-direction the Hall conductivity σ_{xy} .

1.2 Quantum Hall Effect in a 2DEG

In the 2DEG a quasi-free electron is described by the Schrödinger equation. When a magnetic field described by a vector potential $\vec{A} = (-By, 0)$ is added to the system, we introduce the canonical momentum $\vec{\pi}$, which replaces the momentum operator in the Schrödinger equation.

$$\vec{p} \rightarrow \vec{\pi} = \vec{p} - \frac{-e}{c} \vec{A}(\vec{r}) \quad (1)$$

$$\frac{1}{2m_e} \vec{\pi}^2 \psi(\vec{r}) = E \psi(\vec{r}) \quad (2)$$

At this point we define two variables to shorten all the following calculations, the magnetic length l_B and the cyclotron frequency ω_c :

$$l_B = \sqrt{\frac{\hbar c}{eB}} \quad (3)$$

$$\omega_c = \frac{eB}{m_e} \quad (4)$$

Next we define the two operators a, a^\dagger to show that it is possible to write the Hamilton operator in a magnetic field isomorphic to the one which describes a harmonic oscillator:

$$a = \frac{1}{\sqrt{2}} \frac{l_B}{\hbar} (\pi_x - i\pi_y) \quad (5)$$

$$a^\dagger = \frac{1}{\sqrt{2}} \frac{l_B}{\hbar} (\pi_x + i\pi_y) \quad (6)$$

Because these operators should be ladder operators, they gained their form by fulfilling the condition that $[a, a^\dagger] = 1$. So now we can write the eigenvalue equation in the form of a harmonic oscillator:

$$\hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right) \psi(\vec{r}) = E\psi(\vec{r}) \quad (7)$$

$a^\dagger a$ is a number operator, hence we gain the eigenenergies of the electrons in a 2DEG:

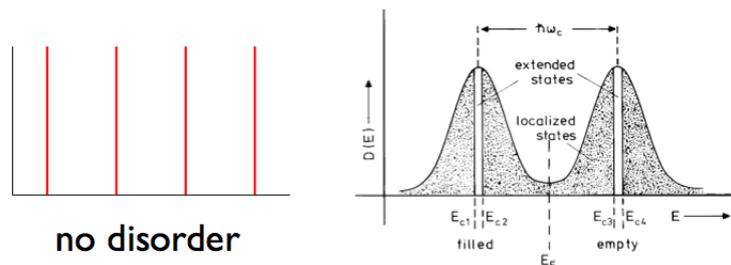
$$E_n = \hbar\omega_c \left(n + \frac{1}{2} \right) \quad (8)$$

These discrete energy levels are called *Landau levels*. Here it is important to remark that the lowest Landau level has the energy $\frac{1}{2}\hbar\omega_c$, it is not zero! Without a magnetic field the energy was given by $E = \frac{\hbar^2 k^2}{2m_e}$, so that we can see that in a magnetic field the energy becomes independent from k . The number of states will not change when a magnetic field is switched on, so there has to be a high degeneracy in each Landau level. To calculate the degeneracy imagine that an electron moves in a cycle in the magnetic field and so every electron needs some space. The radius of this circle is then given by l_B . The degeneracy N_S of each Landau level is then the ratio between the size of the sample A and the area each e^- circle needs

$$N_S = \frac{A}{2\pi l_B^2} = \frac{\Phi}{\Phi_0} \quad (9)$$

where Φ_0 is the flux quantum and is given by $\Phi_0 = hc/e$.

In a perfect sample the energy levels would be discrete as described above. But in a real sample this is not correct because of disorder (impurities, edge states,...), so the Landau levels will broaden and a separation in localized and extended states will occur. Only the extended states will contribute to the current in diagonal and Hall direction, the localized states are bound to the places of disorder and therefore can't move. The graphic



shows the difference between an ideal sample and a real sample. The distance between

two Landau levels is $\hbar\omega_c$ and so it can be changed by changing the magnetic field. By increasing the magnetic field the Landau levels will cross the Fermi energy at some time. It is also possible to increase the density of electrons by increasing the gate voltage. Hence the Fermi energy also crosses at a special point the Landau levels.

Is the Fermi energy in an area of localized states and is changed relatively to the Landau levels by changing the magnetic field or the gate voltage, the conductivity in diagonal direction will not change because the number of the electrons contributing to the current does not change. New incoming electrons into the sample then only will occupy localized states and the conductivity σ_{xx} is zero. The Hall conductivity σ_{xy} then has a plateau while only localized states will become occupied, no new electrons will contribute to the current in y-direction. Indeed when the Fermi energy lies in an area of extended states and its position is changed, the number of occupied extended states does change. The diagonal conductivity then has a value which is not zero because the electrons can use unoccupied states in the Landau level to move. Because of the changing number of occupied extended states the Hall conductivity σ_{xy} does change.

→ Hence in a diagram where the Hall conductivity is printed depending on the magnetic field, plateaus will occur while the Fermi energy is in an area of localized states.

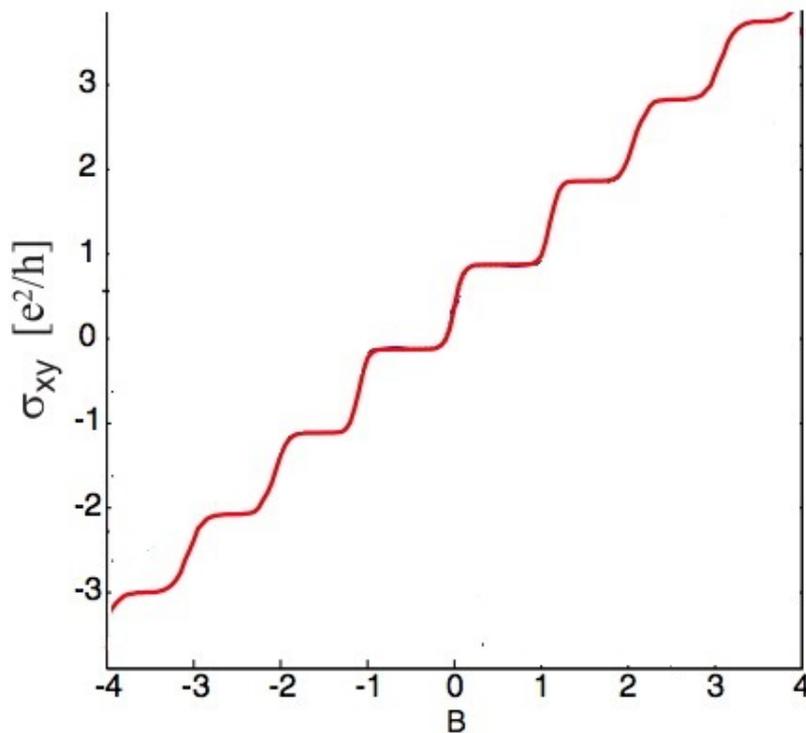


Abbildung 1: Hall conductivity σ_{xy} in a 2DEG, B in arbitrary units

2 QHE in graphene

For graphene we need an other definition of ω_c :

$$\omega_c = \sqrt{2} \frac{v_F}{l_B} \quad (10)$$

This definition comes out of dimensional analysis. For describing the electrons in graphene we need the Dirac equation because of the linear disperion at the two Dirac points:

$$v_F \vec{(\sigma \cdot \vec{p})} \psi(\vec{r}) = E \psi(\vec{r}) \quad (11)$$

When a megnetic field is added perpendicular to the graphene plane, the momentum operator has to be replaced as described in equation (1) and we obtain:

$$-v_F \left(\sigma_x \left(i \partial_x - \frac{e}{c} (-By) \right) + i \sigma_y \partial_y \right) \psi(\vec{r}) = E \psi(\vec{r}) \quad (12)$$

The movement of the electrons in x-direction is not influenced by the magnetic field so they move like free electrons in this direction and we can separate the electron wave function in x- and y-direction with a free wave funtion in x-direction:

$$\psi(x,y) = e^{ikx} \phi(y) \quad (13)$$

By defining $\xi = y/l_B - l_B k$ and using the definition of ω_c we obtain the hamilton (15) operator for the eigenvalue equation (14):

$$\hat{H} \phi(\xi) = -\frac{E\sqrt{2}}{\omega_c} \phi(\xi) \quad (14)$$

$$\hat{H} = \begin{pmatrix} 0 & \xi + \partial_\xi \\ \xi - \partial_\xi & 0 \end{pmatrix} \quad (15)$$

This hamilton operator looks like it can be written with to operators O, O^\dagger for $\xi \pm \partial_\xi$. These two operators shall fulfil the commutator relation of ladder operators $[O, O^\dagger] = 1$ and therefore the operators look like

$$O = \frac{1}{\sqrt{2}}(\xi + \partial_\xi), \quad O^\dagger = \frac{1}{\sqrt{2}}(\xi - \partial_\xi) \quad (16)$$

So we have the following eigenvalue equation:

$$\begin{pmatrix} 0 & O \\ O^\dagger & 0 \end{pmatrix} \phi(\xi) = -\frac{E}{\omega_c} \phi(\xi) \quad (17)$$

where $\phi(\xi)$ is a 2 component vector $\phi(\xi) = (\phi_A(\xi), \phi_B(\xi))^T$ for the two sublattices A and B. For a h.o.-like hamilton operator we need the product $O^\dagger O$ and therefore we write the eigenvalue equation for H^2 and we will obtain the following hamilton operator

$$\hat{H}^2 = \omega_c^2 \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O^\dagger O \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \quad (18)$$

The upper line in here describes the sublattice A and the lower line the sublattice B. The fact that $O^\dagger O$ is a number operator makes it easy to calculate the eigenenergies for the two sublattices.

$$E^2 = (\hbar\omega_c)^2 \cdot (m+1) \quad m = 0,1,2,\dots \quad (19)$$

$$E^2 = (\hbar\omega_c)^2 \cdot n \quad n = 0,1,2,\dots \quad (20)$$

The two sublattices have the same energy and hence the two quantum numbers n, m are not independent and we set $m = n - 1$. At this point every solution of the electron wave function can be written as a two component vector for the two sublattices where each component is a solution $\psi(\xi)$ of a harmonic oscillator.

$$\phi_{N,\mp}(\xi) = \begin{pmatrix} \psi_{N-1}(\xi) \\ \mp\psi_N(\xi) \end{pmatrix} \quad (21)$$

For $n = 0$ exists a zero energy solution, because the energy is given in eq. (19) by $E = \hbar\omega_c\sqrt{n}$. The eigenvalue equation then is written as

$$\begin{pmatrix} 0 & O \\ O^\dagger & 0 \end{pmatrix} \begin{pmatrix} \phi_A(\xi) \\ \phi_B(\xi) \end{pmatrix} = 0 \quad (22)$$

and the two conditions in (23) have to be fulfilled for the zero energy state. Therefore ϕ_B has to be the lowest level solution of the h.o. for $n=0$ and ϕ_A has to be zero!

$$O\phi_B(\xi) = 0 \quad O^\dagger\phi_A(\xi) = 0 \quad (23)$$

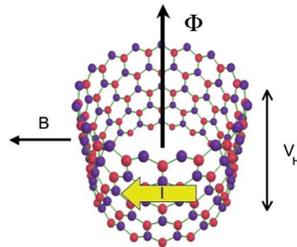
With this the zero energy solution is given by

$$\phi_0(\xi) = \begin{pmatrix} 0 \\ \psi_{B,n=0}(\xi) \end{pmatrix} \quad (24)$$

This zero energy solution described here is very important for the understanding of the anomalous quantum Hall effect in graphene!

2.1 Explaining the QHE in graphene by a Gedankenexperiment

Imagine we would have a cylinder made of graphene (carbon nano tube) where the current is running in a circle on the surface of the cylinder (x-direction) and an external magnetic field, always perpendicular to the surface of the cylinder, is added. The Lorentz force causes a Hall voltage parallel to the middle axis of the cylinder (y-direction). The circling current generates a magnetic flux in y-direction. If the flux can be changed, i.e. with



a solenoid on the middle of the cylinder, only the extended states in the cylinder will be influenced by this change in external flux. The external flux shall not influence the magnetic field. For the extended states the following condition has to be fulfilled:

$$\phi(x + 2\pi R, y) = \phi(x, y) \quad (25)$$

Because of gauge invariance only the phase of the vector potential \vec{A} will change, hence the flux only can change by a flux quantum Φ_0 times an integer n :

$$\Delta\Phi = n \cdot \frac{hc}{e} \quad (26)$$

By changing the flux in the solenoid an integer number of states will enter the cylinder and leave it on the other side. Here we have nearly the same argumentation than in the 2DEG: When the Fermi energy is in an area of localized states, the change of flux will not cause a change in Hall conductivity because the number of occupied extended states remains the same.

Now we want to calculate the number of states which are entering/leaving the cylinder, when the external flux is changed by a flux quantum Φ_0 . Therefore we describe the current in the cylinder by

$$I = c \frac{\delta E}{\delta \Phi} \quad (27)$$

where E is the total energy in the system. Each Landau level contributes to the energy of the current by on states times its degeneracy g . For graphene $g = 4$ because of the two spin states times the two Dirac points. So the change of energy in the system is given by (N : # occupied Landau levels; eV_H : energy each electron has in y-direction)

$$\delta E = \pm 4NeV_H \quad (28)$$

Hence the change in current is $\delta I = eNe^2V_H/h$ and the Hall conductivity can be calculated:

$$\sigma_{xy} = \frac{I}{V_H} = 4N \frac{e^2}{h} \quad (29)$$

This equation is nearly the same than for a 2DEG (there only would be a factor of 2 because of the missing degeneracy from the Dirac points) and would predict a plateau for $N = 0$ and therefore for the zero energy state. But this is not possible as discussed before: Graphene has a zero energy Landau level and hence the conductivity is changing when the Fermi energy moves relatively to the Landau levels. The problem is that the zero energy state has only half the degeneracy than all the other states (see eq. (24)). The degeneracy then can be written as $4N + 2$ to fulfil these conditions.

$$\sigma_{xy} = 2 \frac{e^2}{h} (2N + 1) \quad (30)$$

In measured data this looks then like a shift of $\pm 1/2$ and really was measured in 2005 by Novoselov, Geim, Morosov, et al. and is shown in figure 2.

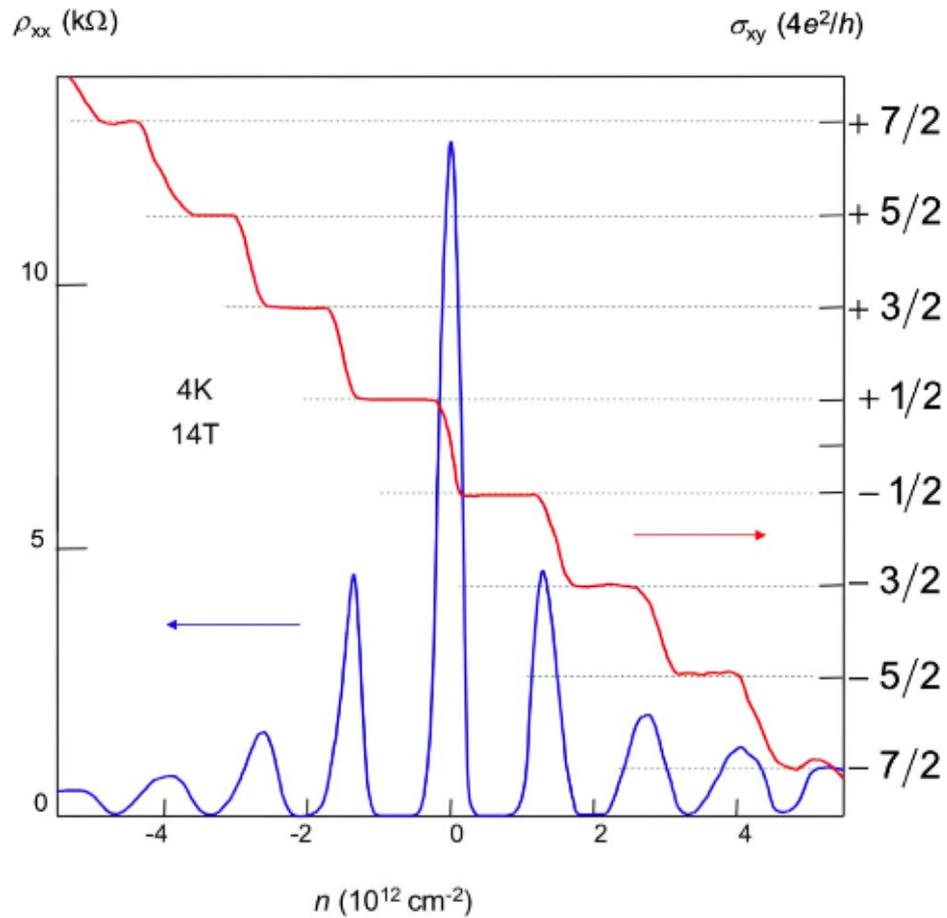


Abbildung 2: anomalous Quantum Hall Effect in graphene

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